Covariant Schrödinger operator

2002 J. Phys. A: Math. Gen. 35 8407

(http://iopscience.iop.org/0305-4470/35/40/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 8407–8434 PII: S0305-4470(02)34731-0

Covariant Schrödinger operator

Josef Janyška[1] **and Marco Modugno**[2]

[1] Department of Mathematics, Masaryk University, Janáčkovo nám 2a, 662 95 Brno, Czech Republic
[2] Department of Applied Mathematics, Florence University, Via S Marta 3, 50139 Florence, Italy

E-mail: janyska@math.muni.cz and modugno@dma.unifi.it

Received 11 March 2002
Published 24 September 2002
Online at stacks.iop.org/JPhysA/35/8407

Abstract
We analyse the Schrödinger operator for a quantum scalar particle in a curved spacetime which is fibred over absolute time and is equipped with given spacelike metric, gravitational field and electromagnetic field. We approach the Schrödinger operator in three independent ways: in terms of covariant differentials induced by the quantum connection, via a quantum Lagrangian and directly by the only requirement of general covariance. In particular, in the flat case, our Schrödinger operator coincides with the standard one.

PACS number: 03.65.Pm
Mathematics Subject Classification: 81Q99, 81S10, 83C99, 18A23, 58A32

Introduction

Since the very beginning of quantum mechanics, the Schrödinger operator has been approached in several ways and investigated in many respects; the related literature is extremely large. Indeed, the Schrödinger equation is one of the greatest successes of physics, hence it should be taken as a touchstone for any possible variation on this subject.

Certainly, the most usual approaches are based on Hamiltonian techniques related to the quantization of the classical Hamiltonian and the standard formalism is basically analytical.

Among the geometric Hamiltonian approaches, we must mention 'geometric quantization' [1, 21, 44, 90, 91, 100]. In this context one can achieve a Schrödinger operator (which includes spacelike scalar curvature) for quantum systems admitting a suitable polarization.

The Feynman path integral is another possible method of achieving the Schrödinger operator (see, for instance, [39, 20]). However, in spite of the fundamental importance of this formalism, its formal theoretical troubles are well known.

Tulczjew [94] achieved the Schrödinger operator by considering a five-dimensional spacetime.

Possible modifications of the Schrödinger operator have been investigated as well. Possible nonlinear variations have been discussed by Döbner and Goldin *et al* [24–30] in the framework of nonlinear quantum mechanics. A modification of the quantum dynamics has been proposed by Ghirardi [43] in view of a unified picture of microscopic and macroscopic systems.

Another interesting aspect of the Schrödinger operator concerns its transformation rule with respect to the change of frame of reference and coordinates [89].

An even deeper problem would be a formulation of the Schrödinger operator on a curved spacetime and an analysis of its relation with the general principle of relativity and the principle of equivalence. A true solution of this problem is today too ambitious because it would involve well established quantum gravity and quantum field theory on a curved spacetime.

Among the attempts to formulate a quantum theory in the framework of general relativity, we quote Prugovečki [84, 85]. On the other hand, several authors have discussed the problem of the equivalence principle for a quantum system (see, for instance, Camacho [9–16]) and analysed the behaviour of a quantum system with respect to accelerated observers (see, for instance, Bialynicki-Birula, Kalinski, Mashhoon and others [4, 5, 62, 77, 86]).

A more limited, but interesting, goal is the formulation of the Schrödinger operator on a curved spacetime fibred over absolute time and equipped with a spacelike Riemannian metric, which will be referred to as 'Galilei curved spacetime'. The idea of such a spacetime goes back to Cartan [18, 19] and has been further analysed by several authors, including Bargmann [3], Dautcourt [22], Dombrowski and Horneffer [31], Le Bellac and Levy-Leblond [73, 74], Ehlers [37], Havas [45], Mangiarotti [76], Trautman [92, 93]. This model of spacetime is intermediate between the Newton spacetime and the Einstein spacetime. Hence, it provides an intermediate setting between the standard non-relativistic quantum mechanics and a possible general relativistic quantum theory. In this framework we can discuss the behaviour of the quantum system in accelerated frames, but we lose the description of phenomena related to the speed of light and to the Lorentz metric. Among the approaches to the Schrödinger operator in such a curved spacetime, we would like to mention the pioneering paper by DeWitt [23], the paper by Kuchař [65], who approaches the Schrödinger operator via Dirac's constraint method in a coordinate independent way, and the papers by Fanchi [38] and Kyprianidis [71]. The Lagrangian approach in a Bargmann framework due to Duval and Künzle [32–36, 66–70] deserves a special mention.

The standard concepts of 'relativistic' and 'non-relativistic' theory are not adequate for our discussions. We need a more careful and refined use of these words. In this paper, the word 'general (special) relativity' refers both to the Einstein general (special) relativity and to the Galilei general (special) relativity. In fact, both theories are formulated in a way which is independent of the choice of an accelerated (inertial) observer. In both cases, an observer is described by a normalized timelike vector field, but the mathematical definition of the 'timelike' character and of the 'normalization' is different in the two models.

The present paper is aimed at analysing the Schrödinger operator on a curved spacetime in the framework of a covariant formulation of quantum mechanics that here will be referred to as 'covariant quantum mechanics'. This formulation was originally proposed for scalar particles by Jadczyk and Modugno [47–49] and further developed in cooperation with Janyška, Saller, Tejero Prieto and Vitolo [46, 50, 51, 53, 55, 57, 59–61, 80–82, 87, 96, 98, 99]. It has been extended to spin particles in cooperation with Canarutto [17] and partially (up to pre-quantum operators) to a Lorentzian setting in cooperation with Janyška and Vitolo [52, 54, 56, 58, 97]. This approach is based on non-standard geometric methods, such as fibred manifolds, jet spaces, nonlinear connections, systems of connections, cosymplectic structures and Frölicher smooth spaces.

In a few words, the scheme of this model is the following. The classical spacetime is constituted by a manifold fibred over absolute time and equipped with a spacelike Riemannian metric, a time preserving and metric preserving connection (gravitational field) and a closed 2-form (electromagnetic field). As classical phase space we assume the first jet space of spacetime. The above fundamental fields yield, in a covariant way, a second-order connection, hence a covariant formulation of the law of motion. Moreover, they yield, in a covariant way, a global cosymplectic 2-form, which locally induces a gauge-dependent Lagrangian and a gauge- and observer-dependent Hamiltonian. We show a distinguished Lie algebra of quadratic functions (including the Hamiltonian) different from the Poisson Lie algebra. Then, we formulate the covariant quantum theory for a scalar particle, affected by the given gravitational and electromagnetic fields, by considering a complex bundle over spacetime equipped with a Hermitian metric with values in the space of spacelike volume forms and a universal, Hermitian connection, whose curvature is proportional to the cosymplectic 2-form. From these two objects, we derive, in a covariant way, all other quantum objects. In order to get rid of the observers we use a criterion of projectability. In particular, this method yields good candidates for the quantum operators, through the isomorphism between the Lie algebra of Hermitian quantum vector fields and the Lie algebra of projectable quadratic classical functions. Moreover, this method yields a good candidate for the Schrödinger operator, equivalently, via quantum covariant differentials and via a quantum Lagrangian.

Let us summarize the main features of 'covariant quantum mechanics', for a scalar particle, in view of a comparison with more standard formulations.

We are concerned only with 'fundamental fields'. Therefore, we consider only external gravitational and electromagnetic fields for the classical background of the quantum system. In fact, we think that only a fundamental setting has the right to demand a covariance principle and has a chance to produce results possibly interesting for further covariant developments.

Our quantum theory is not intended as a 'quantization procedure' of a classical system into a quantum system. We just propose a direct approach to quantum mechanics. The classical theory is involved essentially through the spacetime structure and not really through the classical particle mechanics. Actually, the classical spacetime plays the role of background space, as it carries the information of the external classical gravitational and electromagnetic fields affecting the quantum particle. More precisely, this information is carried by the classical spacetime itself as the base space of the quantum bundle and by the classical phase space, which here plays the role of the space of classical observers. The main information of the given classical gravitational and electromagnetic fields is encoded in the cosymplectic 2-form of the classical phase space, which plays the role of curvature of the quantum connection.

We take the well established *results* of quantum mechanics, such as the standard Schrödinger equation and quantum operators, as the touchstone of our model. On the other hand, according to the aims of our theory, we disregard those standard *methods* for deriving quantum objects, which are incompatible with general covariance.

Our basic guide is the *covariance* of the theory as a heuristic requirement. Further, we look for manifest covariance. Nowadays, the concept of 'covariance' has been formulated in a rigorous mathematical way through the geometric concept of 'naturality' [63]. Our theory provides explicit expressions of all objects for any *accelerated observer* and yields, at the same time, an interpretation in terms of gravitational field, according to the principle of equivalence.

According to the covariance of the theory, *time* is not just a parameter, but a fundamental object. Moreover, the main objects of the theory are not assumed to be split into time and space components.

As classical *phase space* we take the first jet space of spacetime and not its tangent space; indeed, this minimal choice allows us to skip anholonomic constraints.

Another consequence of our approach is that classical mechanics is ruled by a *cosymplectic structure* [75] and not by a symplectic structure. Actually, a spacelike symplectic structure arises in our model, but it describes only the spacelike aspects of the classical theory and is insufficient to account, in a covariant way, for classical dynamics.

We emphasize the fact that classical mechanics can be formulated in a covariant way by a *second-order connection* and by a *Lagrangian approach*, but not by a Hamiltonian approach, because the Hamiltonian function depends essentially on an observer.

The *Hamiltonian* is not postulated as an additional object of our classical setting, but it can be 'extracted' by an observer, from a local potential of the cosymplectic 2-form.

An achievement of our theory is the Lie algebra of '*special quadratic functions*' (different from the Poisson algebra), which allows us to treat energy, momentum and spacetime functions on the same footing. This algebra and its subalgebras control fully the classical and quantum symmetries.

All objects of quantum mechanics are derived, in a covariant way, from three *minimal objects*. Here, we have some novelties.

The quantum bundle lives *on spacetime* and not on the phase space, and the quantum connection is '*universal*'. These assumptions allow us to skip all problems of polarizations [100]. In a sense, we obtain naturally a covariant polarization and this is sufficient for our purposes. Indeed, we replace the problematic search for such inclusions with a *method of projectability*, which turns out to be our implementation of covariance in the quantum theory.

Another new feature concerns the *Hermitian metric* of the quantum bundle, which takes its values in the space of spacelike volume forms. This assumption allows us to skip the problems related to half-densities. On the other hand, this assumption requires the projectability of Hermitian vector fields. Indeed, this feature turns out to be an opportunity in our approach.

We show the strict relation of the quantum connection with the classical *Poincaré–Cartan form*. In a sense, the quantum connection can be regarded as a global and gauge-independent version of the above local and gauge-dependent form.

The Schrödinger equation is obtained, in a covariant way, both through a *differential* and a *Lagrangian approach*.

The quantum operators arise, in a covariant way, from the *classification* of distinguished first- and second-order differential operators of the quantum framework and not from a quantization requirement of a classical system.

The seat for the covariant probabilistic interpretation of quantum mechanics is the *Hilbert bundle* over time, which arises naturally from the quantum bundle. This bundle is observer independent and each observer splits it into time and a Hilbert space.

We do not postulate transition maps for the classical and quantum theory (concerning transition of coordinates, potentials, etc), but we start from simple intrinsic axioms and derive the transition rules from them. Thus, the groups involved in the theory arise as groups of automorphisms of the original geometric structures.

In a few words, we start with really minimal geometric structures representing fundamental physical fields and proceed along a thread naturally imposed by the only requirement of general covariance.

In the flat case, the results of our model fit exactly the corresponding results of standard quantum mechanics. Thus, in the flat case, we do need to exhibit successful examples, as is sometimes required in alternative theories.

The specific aim of the present paper is to prove that, under weak assumptions on the classical background and minimal assumptions on the quantum bundle, the (second-order) Schrödinger operator is determined by the only requirements of covariance and invariance with respect to timescales. Under these hypotheses, the Schrödinger operator turns out to be linear and of Lagrangian type.

Thus, we achieve the Schrödinger operator in three independent covariant ways:

- In terms of covariant differential and codifferential induced by the quantum connection, by using a criterion of projectability (hence the independence from observers).
- In terms of a quantum Lagrangian associated with the quantum connection and the quantum metric, by using a criterion of projectability (hence the independence from observers).
- By postulating just the covariance of the operator (this approach also yields the uniqueness).

In the first instance, our approach to the Schrödinger operator has no direct relation with Hamiltonian techniques and with the quantization of the classical Hamiltonian. Actually, the Schrödinger operator is global, gauge independent and observer independent, while the classical Hamiltonian is local, gauge dependent and observer dependent. On the other hand, the quantum Hamiltonian operator can be achieved *a posteriori* from the Schrödinger operator by taking into account the pre-quantum operator associated, in a covariant way, with the classical Hamiltonian [46, 49].

The covariance with respect to the timescales plays a role on the same footing of covariance with respect to the change of spacetime coordinates, to the change of observers and to the change of quantum bases.

If we would assume a distinguished timescale, then the requirement of covariance would be compatible with many additional terms. Actually, we could take into account some timescales arising from constants which have a role in our model, such as the Planck constant, the gravitational constant, the mass of the particle, etc. It seems that these timescales yield additional terms which are too big to be physically reasonable. But, we cannot exclude that one could find a reasonable distinguished timescale which breaks the covariance with respect to timescales. We could make similar considerations for length scales; but nothing essentially new would arise, because timescales and length scales are related, in the framework of our model, by $\hbar/m$.

We denote the sheaf of local sections of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and the sheaf of local fibred morphisms between the fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p' : \boldsymbol{F}' \to \boldsymbol{B}$, respectively, by $\sec(\boldsymbol{B}, \boldsymbol{F})$ and $\mathrm{fib}(\boldsymbol{F}, \boldsymbol{F}')$.

1. Classical background

In this section we summarize [46, 49] the basic concepts and results concerning the classical spacetime, in view of the quantum theory. With respect to previous papers, here we add a new discussion on the covariance group bundles, on metric extensions, Galileian connections and classical potentials.

1.1. Units of measurement

Covariance requires a rigorous treatment of units of measurement.

We assume [46, 49] the following 'positive one-dimensional semi-vector spaces' over $\mathbb{R}^+$ as fundamental unit spaces: the space $\mathbb{T}$ of *time intervals*, the space $\mathbb{L}$ of *lengths* and the space $\mathbb{M}$ of *masses*. Moreover, we assume the *Planck constant* $\hbar \in \mathbb{T}^* \otimes \mathbb{L}^2 \otimes \mathbb{M}$.

We refer to particles with mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^* \otimes \mathbb{L}^{\frac{3}{2}} \otimes \mathbb{M}^{\frac{1}{2}}$.

We define a *time unit* to be an element $u_0 \in \mathbb{T}$ or its dual $u^0 \in \mathbb{T}^*$.

1.2. Classical spacetime

We start with our postulate concerning the classical curved Galileian spacetime.

As the first *postulate* of our classical model, we assume *time* to be an affine space $\boldsymbol{T}$ associated with the vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes \mathbb{R}$. Moreover, we assume *spacetime* to be an oriented (3 + 1)-dimensional manifold $\boldsymbol{E}$ fibred over time by the *time map* $t : \boldsymbol{E} \to \boldsymbol{T}$.

Thus, the time fibring yields the *time form* $\mathrm{d}t : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$.

We define the *spacetime charts* to be charts $(x^\lambda) \equiv (x^0, x^i)$ of spacetime, which are adapted to the time fibring, the affine structure of time, the orientation of time and spacetime, and to a time unit u_0.

We shall always refer to spacetime charts.

If (x^λ) is a spacetime chart, then the induced local bases of $T\boldsymbol{E}$, $V\boldsymbol{E}$, $T^*\boldsymbol{E}$ and $V^*\boldsymbol{E}$ are denoted, respectively, by (∂_λ), (∂_i), (d^λ) and $(\check{d}^i)$.

The coordinate expression of the time form is $\mathrm{d}t = u_0 \otimes d^0$.

If (x^λ) and $(\acute{x}^\lambda)$ are two spacetime charts, then we set

$$\sigma^\lambda_{\mu_1\ldots\mu_r} := \acute{\partial}_{\mu_1} \ldots \acute{\partial}_{\mu_r} x^\lambda \qquad \acute{\sigma}^\lambda_{\mu_1\ldots\mu_r} := \partial_{\mu_1} \ldots \partial_{\mu_r} \acute{x}^\lambda$$

and obtain $\acute{x}^0 = \acute{\sigma}^0_0 x^0 + \acute{\sigma}^0$, with $\acute{\sigma}^0 \in \mathbb{R}$, $\acute{\sigma}^0_0 \in \mathbb{R}^+$, $\acute{\sigma}^0_i = 0$, $\det\left(\acute{\sigma}^i_j\right) > 0$.

We define the *timescale covariance bundle* to be the trivial bundle $\mathcal{G}(\mathbb{T}) := \boldsymbol{E} \times \mathbb{R}^+$ of $\mathbb{R}^+$-linear automorphisms of $\mathbb{T}$.

We define

- a *time automorphism* to be an orientation preserving affine isomorphism $\underline{\phi} : \boldsymbol{T} \to \boldsymbol{T}$;
- a *spacetime automorphism* to be a local orientation preserving fibred diffeomorphism $\phi : \boldsymbol{E} \to \boldsymbol{E}$ over a time automorphism $\underline{\phi} : \boldsymbol{T} \to \boldsymbol{T}$.

The coordinate expression of a spacetime automorphism ϕ is of the type

$$\phi^0 = a^0_0 x^0 + a^0 \qquad \phi^i \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$$

with $a^0_0 \in \mathbb{R}^+$, $a^0 \in \mathbb{R}$, $\det(\partial_j \phi^i) > 0$.

If ϕ is a spacetime automorphism, then we set $\phi^\lambda_{\mu_1\ldots\mu_r} := \partial_{\mu_1} \ldots \partial_{\mu_r} \phi^\lambda$.

If $x \equiv (x^\lambda)$ and $\acute{x} \equiv (\acute{x}^\lambda)$ are two spacetime charts, then we obtain the spacetime automorphism $\phi := x^{-1} \circ \acute{x}$. Hence, we obtain $x \circ \phi \circ x^{-1} = \acute{x} \circ x^{-1}$, which yields $\phi^\lambda_{\mu_1\ldots\mu_r} = \acute{\sigma}^\lambda_{\mu_1\ldots\mu_r}$.

Let $\mathrm{aut}(\boldsymbol{E}) \subset \mathrm{fib}(\boldsymbol{E}, \boldsymbol{E})$ be the sheaf of spacetime automorphisms. Then, for each $s \geqslant 0$ and $e \in \boldsymbol{E}$, we define the set

$$\mathcal{G}^s_e(\boldsymbol{E}) := \{(j_s\phi)(e) \mid \phi \in \mathrm{aut}(\boldsymbol{E}), \phi(e) = e\}.$$

For each integer $s \geqslant 0$, we define the *spacetime covariance bundle*, of order s, to be the bundle

$$\mathcal{G}^s(\boldsymbol{E}) := \bigsqcup_{e\in\boldsymbol{E}} \mathcal{G}^s_e(\boldsymbol{E}) \to \boldsymbol{E}. \tag{1.1}$$

Clearly, the fibre of $\mathcal{G}^0(\boldsymbol{E})$ has dimension 0. We denote the fibred chart of $\mathcal{G}^s(\boldsymbol{E})$ induced by a spacetime chart (x^λ) by $\left(x^\lambda; \mathrm{g}^0_0, \mathrm{g}^i_\lambda; \ldots; \mathrm{g}^i_{\lambda_1\ldots\lambda_s}\right)$.

The bundle $\mathcal{G}^s(\boldsymbol{E})$ turns out a group bundle through the composition fibred morphism over $\boldsymbol{E}$:

$$\begin{aligned} \mu^s &: \mathcal{G}^s(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathcal{G}^s(\boldsymbol{E}) \to \mathcal{G}^s(\boldsymbol{E}) \\ &: \left((j_s \overset{1}{\phi})(e), (j_s \overset{2}{\phi})(e)\right) \mapsto \left(j_s(\overset{1}{\phi} \circ \overset{2}{\phi})\right)(e). \end{aligned}$$

1.3. Observers

Then, we introduce the notion of an observer and the associated transition maps.

An *observer* is defined to be a section $o \in \sec(\boldsymbol{E}, J_1\boldsymbol{E})$, i.e. a section

$$o \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E}) \quad \text{which projects on} \quad \mathbf{1} \in \mathbb{T}^* \otimes \mathbb{T}. \tag{1.2}$$

The coordinate expression of an observer is of the type $o = u^0 \otimes \left(\partial_0 + o^i_0 \partial_i\right)$, with $o^i_0 \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$.

The charts (x^λ) for which $o^i_0 = 0$ are said to be *adapted* to o. Each chart (x^λ) determines the observer $o := u^0 \otimes \partial_0$.

Each observer o yields the *spacelike projection* $T\boldsymbol{E} \to V\boldsymbol{E}$ through the map

$$\nu[o] = \left(d^i - o^i_0 d^0\right) \otimes \partial_i \in \sec(\boldsymbol{E}, T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}). \tag{1.3}$$

If o and $\acute{o}$ are two observers, then we can write $\acute{o} = o + v$, with $v \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})$.

The Abelian group bundle $\mathcal{T}(\boldsymbol{E}) := \mathbb{T}^* \otimes V\boldsymbol{E} \to \boldsymbol{E}$ is called the *observer transition bundle*.

Its sections will yield the *transition maps* for 'observed objects'.

1.4. Metric field

Then, we discuss the metric structure of spacetime.

We define a *spacelike metric* to be a scaled Riemannian metric $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes S^2 V^*\boldsymbol{E}$ of the fibres of $\boldsymbol{E}$.

With reference to a mass $m \in \mathbb{M}$, we define the *re-scaled spacelike metric*

$$G := \frac{m}{\hbar} g = G^0_{ij} u_0 \otimes \check{d}^i \otimes \check{d}^j : \boldsymbol{E} \to \mathbb{T} \otimes S^2 V^*\boldsymbol{E}. \tag{1.4}$$

We denote the contravariant spacelike metric and the contravariant re-scaled spacelike metric by $\bar{g} : \boldsymbol{E} \to \mathbb{L}^{2*} \otimes S^2 V\boldsymbol{E}$ and $\bar{G} := \frac{\hbar}{m}\bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes S^2 V\boldsymbol{E}$.

We denote the natural bundle of re-scaled spacelike metrics by $\operatorname{Met}(\boldsymbol{E}) \subset \mathbb{T} \otimes S^2 V^*\boldsymbol{E}$.

As the second *postulate* of our classical model, we assume a spacelike metric g.

The spacelike metric g and the spacetime orientation naturally yield the scaled *spacelike volume form* and the *spacetime volume form*

$$\eta = \sqrt{|g|}\check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3 : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E} \tag{1.5}$$

$$\upsilon := \mathrm{d}t \wedge \eta = \sqrt{|g|} u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^*\boldsymbol{E}. \tag{1.6}$$

We set $\upsilon^0 := \upsilon(u^0) = \sqrt{|g|} d^0 \wedge d^1 \wedge d^2 \wedge d^3$.

For each projectable vector field X of $\boldsymbol{E}$, we define the *timelike divergence* and the *spacelike divergence*, respectively

$$\operatorname{div}_{\mathrm{dt}} X \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \quad \text{by the equality} \quad L[X]\,\mathrm{d}t = (\operatorname{div}_{\mathrm{dt}} X)\,\mathrm{d}t$$

$$\operatorname{div}_\eta X \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \quad \text{by the equality} \quad L[X]\eta = (\operatorname{div}_\eta X)\eta.$$

For each vector field X of $\boldsymbol{E}$, we define the *spacetime divergence*

$$\operatorname{div}_\upsilon X \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \quad \text{by the equality} \quad L[X]\upsilon = (\operatorname{div}_\upsilon X)\upsilon.$$

We have the coordinate expressions

$$\operatorname{div}_{\mathrm{dt}} X = \partial_0 X^0 \qquad \operatorname{div}_\eta X = X^0 \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}}$$

$$\operatorname{div}_\upsilon X = \frac{\partial_\lambda(X^\lambda\sqrt{|g|})}{\sqrt{|g|}}.$$

The re-scaled spacelike metric G naturally yields

- the fibre-wise Riemannian connection $\varkappa[G] : V\boldsymbol{E} \to V^*\boldsymbol{E} \underset{V\boldsymbol{E}}{\otimes} VV\boldsymbol{E}$
- the fibre-wise curvature tensor $R[\varkappa] : V\boldsymbol{E} \to \Lambda^2 V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}$
- the fibre-wise Ricci tensor $\mathrm{Ricci}[\varkappa] := C^1_1 R[\varkappa] : \boldsymbol{E} \to V^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V^*\boldsymbol{E}$
- the fibre-wise scaled scalar curvature $r[G] := \langle \bar{G}, \mathrm{Ricci}[\varkappa]\rangle : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R}$.

1.5. Metric extensions

The extensions of the spacelike metric to spacetime metrics will play an interesting role in understanding the Galileian connections and their potentials.

A section $\widetilde{G} \in \sec(\boldsymbol{E}, \mathbb{T} \otimes S^2T^*\boldsymbol{E})$, whose vertical restriction is G, is called a *metric extension*.

The coordinate expression of a metric extension is of the type

$$\widetilde{G} = u_0 \otimes \left(G^0_{ij} d^i \otimes d^j + G^0_{0j} d^0 \otimes d^j + G^0_{i0} d^i \otimes d^0 + G^0_{00} d^0 \otimes d^0\right) \tag{1.7}$$

where $G^0_{ij} \equiv G(\partial_i, \partial_j)$ and $G^0_{0j}, G^0_{i0}, G^0_{00} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ with $G^0_{0i} = G^0_{i0}$.

In particular, each observer o yields the metric extension $\widetilde{G}[o] := \nu^*[o]G$, whose expression, in an adapted chart, is $\widetilde{G}[o] = G^0_{ij} u_0 \otimes d^i \otimes d^j$.

Let us consider a metric extension $\widetilde{G}$ and an observer o, and refer to an adapted chart. Then, we obtain the section

$$\begin{aligned} A[\widetilde{G}, o] &:= o \lrcorner \widetilde{G} - \tfrac{1}{2} o \lrcorner o \lrcorner \widetilde{G} \\ &= \tfrac{1}{2} G^0_{00} d^0 + G^0_{i0} d^i \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}). \end{aligned} \tag{1.8}$$

Moreover, we obtain $\widetilde{G} = \nu^*[o]G + \mathrm{d}t \otimes A[\widetilde{G}, o] + A[\widetilde{G}, o] \otimes \mathrm{d}t$. Clearly, we have $o \lrcorner A[\widetilde{G}, o] = \frac{1}{2} o \lrcorner o \lrcorner \widetilde{G}$.

Let us consider a metric extension $\widetilde{G}$ and two observers $o, \acute{o} + v$, and refer to a chart adapted to $\acute{o}$. Then, we obtain

$$\begin{aligned} A[\widetilde{G}, \acute{o}] - A[\widetilde{G}, o] &= -\tfrac{1}{2} G(v, v) + \nu[o] \lrcorner G^\flat(v) \\ &= \tfrac{1}{2} G^0_{ij} v^i_0 v^j_0 d^0 + G^0_{ij} v^j_0 d^i. \end{aligned} \tag{1.9}$$

Let us consider a form $A \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ and an observer o, and refer to an adapted chart. Then, we obtain the metric extension

$$\widetilde{G}[A, o] := \nu^*[o]G + \mathrm{d}t \otimes A + A \otimes \mathrm{d}t. \tag{1.10}$$

Moreover, we obtain $A[\widetilde{G}[A, o], o] = A$. Clearly, we have $\frac{1}{2} o \lrcorner o \lrcorner \widetilde{G}[A, o] = o \lrcorner A$.

Let us consider an observer o. Then, the maps $\widetilde{G} \mapsto A[\widetilde{G}, o]$ and $A \mapsto \widetilde{G}[A, o]$ are inverse bijections.

1.6. Galileian connections

Galileian connections are fundamental objects of our theory.

We define a *spacetime connection* to be a connection K of the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$, which is linear, torsion free and such that $\nabla\, \mathrm{d}t = 0$.

A spacetime connection K is said to be *metric* if $\nabla G = 0$.

A metric spacetime connection K is said to be *Galileian* if its curvature tensor fulfils the condition $R_{0\lambda}{}^{j}{}^{i}{}_{\mu} = R_{0\mu}{}^{i}{}^{j}{}_{\lambda}$.

The coordinate expression of a spacetime connection is of the type $K = d^\lambda \otimes \partial_\lambda + K_\lambda{}^\mu{}_\nu \dot{x}^\nu d^\lambda \otimes \dot{\partial}_\mu$, with $K_\lambda{}^0{}_\nu = 0$ and $K_\mu{}^i{}_\nu = K_\nu{}^i{}_\mu$. The spacetime connections are the sections of a second-order natural bundle $\mathrm{Con}(\boldsymbol{E}) \to \boldsymbol{E}$.

If K is a spacetime connection, o an observer and we refer to an adapted chart, then we obtain

$$\nabla o = -K_\lambda{}^i{}_0 u^0 \otimes d^\lambda \otimes \partial_i \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes (T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E})) \tag{1.11}$$

$$\Phi[K, o] := \mathrm{Ant}(\nu^*[o](G^\flat(\nabla o))) = -K_{\lambda}{}^0_{j}{}_0\, d^\lambda \wedge d^j \in \sec(\boldsymbol{E}, \Lambda^2 T^*\boldsymbol{E}) \tag{1.12}$$

where Ant is the antisymmetrization operator and $K_{\lambda}{}^0_{j}{}_0 := G^0_{ij} K_\lambda{}^i{}_0$.

If o is an observer, then there is a unique metric spacetime connection $K[o]$, such that $\Phi[K, o] = 0$. Indeed, $K[o]$ is given by

$$K[o] = o_\tau + \nu^*[o_\tau](\varkappa) - \mathrm{Sym}(G^{\sharp 1}(\nu^*[o](L[o]G))) \tag{1.13}$$

where o_τ is the tangent prolongation of the observer, $\nu^*[o_\tau](\varkappa)$ the pullback of $\varkappa$ induced by the observer, $\nu^*[o](L[o]G)$ the pullback of $L[o]G$ induced by the observer and $G^{\sharp 1}$ the musical morphism applied to the first component of the tensor $\nu^*[o](L[o]G)$.

The coordinate expression of $K[o]$, in an adapted chart, is [60]

$$\begin{aligned} &K_0{}^i{}_0 = 0 \\ &K_h{}^i{}_0 = K_0{}^i{}_h = -\tfrac{1}{2} G^{ij}_0 \partial_0 G^0_{jh} \\ &K_h{}^i{}_k = K_k{}^i{}_h = -\tfrac{1}{2} G^{ij}_0 \left(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}\right). \end{aligned} \tag{1.14}$$

For each 2-form $\Phi \in \sec(\boldsymbol{E}, \Lambda^2 T^*\boldsymbol{E})$, we set

$$\widehat{\Phi} := G^{\sharp 1}(\Phi) = \Phi^i_{0\mu} u^0 \otimes \partial_i \otimes d^\mu \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes (V\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} T^*\boldsymbol{E})).$$

Let us consider an observer o and refer to an adapted chart.

(1) If K is a metric spacetime connection, then we have

$$K = K[o] + \mathrm{d}t \otimes \widehat{\Phi}[K, o] + \widehat{\Phi}[K, o] \otimes \mathrm{d}t. \tag{1.15}$$

(2) If $\Phi \in \sec(\boldsymbol{E}, \Lambda^2 T^*\boldsymbol{E})$, then we obtain the metric spacetime connection

$$K[o, \Phi] := K[o] + \mathrm{d}t \otimes \widehat{\Phi} + \widehat{\Phi} \otimes \mathrm{d}t. \tag{1.16}$$

The coordinate expression of the above connections K is

$$\begin{aligned} &K_0{}^i{}_0 = -G^{ij}_0 2\Phi_{oj} \\ &K_0{}^i{}_h = K_h{}^i{}_0 = -\tfrac{1}{2} G^{ij}_0 \left(2\Phi_{hj} + \partial_0 G^0_{hj}\right) \\ &K_k{}^i{}_h = K_h{}^i{}_k = -\tfrac{1}{2} G^{ij}_0 \left(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}\right). \end{aligned} \tag{1.17}$$

Thus, an observer yields a local bijection between metric spacetime connections and spacetime 2-forms.

Let K be a metric spacetime connection and let us refer to an observer o. Then [46, 60],

$$K \text{ is Galileian} \quad \Leftrightarrow \quad \mathrm{d}\Phi[K, o] = 0. \tag{1.18}$$

If K is a Galileian connection and o an observer, then each local potential $A[K, o] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ of $\Phi[K, o]$ is called a *potential* of K.

The potential $A[K, o]$ is locally defined up to a gauge of the type $\mathrm{d}f$, with $f \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

Let us consider an observer o and refer to an adapted chart.

(1) If K is a Galileian connection, then we have

$$K = K[o] + \mathrm{d}t \otimes \widehat{\mathrm{d}A}[K, o] + \widehat{\mathrm{d}A}[K, o] \otimes \mathrm{d}t. \tag{1.19}$$

(2) If $A \in \mathrm{sec}(\boldsymbol{E}, T^*\boldsymbol{E})$, then we obtain the Galileian connection

$$K[o, A] := K[o] + \mathrm{d}t \otimes \widehat{\mathrm{d}A} + \widehat{\mathrm{d}A} \otimes \mathrm{d}t. \tag{1.20}$$

The coordinate expression of the above connections K is

$$\begin{aligned} K_0{}^i{}_0 &= -G^{ij}_0(\partial_0 A_j - \partial_j A_0) \\ K_0{}^i{}_h &= K_h{}^i{}_0 = -\tfrac{1}{2}G^{ij}_0\left(\partial_h A_j - \partial_j A_h + \partial_0 G^0_{hj}\right) \\ K_k{}^i{}_h &= K_h{}^i{}_k = -\tfrac{1}{2}G^{ij}_0\left(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}\right) \end{aligned} \tag{1.21}$$

i.e.

$$K_\lambda{}^i{}_\mu = -\tfrac{1}{2}G^{ij}_0\left(\partial_\lambda G^0_{j\mu} + \partial_\mu G^0_{j\lambda} - \partial_j G^0_{\lambda\mu}\right) \tag{1.22}$$

where we refer to the metric extension $\widetilde{G}[A[o], o]$.

Thus, an observer yields a local bijection between Galileian connections and spacetime 1-forms, modulo differentials of spacetime functions.

1.7. Classical phase space

We briefly sketch the classical phase space and the objects induced on it by a Galileian connection K.

The *phase space* is defined to be the first jet space $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ of spacetime [46, 49, 55, 88].

We denote the fibred charts of $J_1\boldsymbol{E}$ induced by a spacetime chart (x^λ) by $\left(x^\lambda, x^i_0\right)$.

We recall that $J_1\boldsymbol{E} \to \boldsymbol{E}$ is an affine bundle associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$. Hence, the vertical space of $J_1\boldsymbol{E}$ with respect to $\boldsymbol{E}$ turns out to be $V_0 J_1\boldsymbol{E} = \mathbb{T}^* \otimes V\boldsymbol{E}$.

We recall the natural *contact maps*

$$\text{д} = u^0 \otimes \left(\partial_0 + x^i_0\partial_i\right) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \tag{1.23}$$

$$\theta = \left(d^i - x^i_0 d^0\right) \otimes \partial_i : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} V\boldsymbol{E}. \tag{1.24}$$

The metric G and a spacetime connection K yield in a covariant way [46, 49, 60] the following objects, which are global, gauge independent and observer independent:

- the *phase connection*

$$\begin{aligned} &\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\otimes} TJ_1\boldsymbol{E} \\ &\Gamma = d^\lambda \otimes \partial_\lambda + \left(\Gamma_\lambda{}^{i\,0}_{00} + \Gamma_\lambda{}^{i\,0}_{0j} x^j_0\right) d^\lambda \otimes \partial^0_i \end{aligned} \tag{1.25}$$

  where $\Gamma_\lambda{}^{i\,0}_{0\mu} = K_\lambda{}^i{}_\mu$;
- the *phase 2-form*

$$\begin{aligned} &\Omega := \nu[\Gamma] \bar{\wedge} \theta : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E} \\ &\Omega = G^0_{ij}\left(d^i_0 - \left(\Gamma_\lambda{}^{i\,0}_{00} + \Gamma_\lambda{}^{i\,0}_{0j} x^j_0\right) d^\lambda\right) \wedge \theta^j \end{aligned} \tag{1.26}$$

  where $\nu[\Gamma]$ is the vertical-valued form associated with Γ and $\bar{\wedge}$ is the wedge product followed by the contraction through G;

- the *second-order phase connection*

$$\gamma := \text{д}\lrcorner\Gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$$
$$\gamma = u^0 \otimes \left(\partial_0 + x_0^i\partial_i + \gamma_{0}{}^{i}_{0}\partial_i^0\right) \tag{1.27}$$

where $\gamma_{0}{}^{i}_{0} = \Gamma_{h}{}^{i}{}_{0}{}^{0}{}_{k}\, x_0^h\, x_0^k + 2\Gamma_{h}{}^{i}{}_{0}{}^{0}{}_{0}\, x_0^h + \Gamma_{0}{}^{i}{}_{0}{}^{0}{}_{0}$;
- the *phase 2-vector*

$$\Lambda := \check{\Gamma}\bar{\wedge}\nu : J_1\boldsymbol{E} \to \Lambda^2 VJ_1\boldsymbol{E}$$
$$\Lambda = G_0^{ij}\left(\partial_i + \left(\Gamma_{i}{}^{h}{}_{0}{}^{0}{}_{0} + \Gamma_{i}{}^{h}{}_{0}{}^{0}{}_{k}x_0^k\right)\partial_h^0\right) \wedge \partial_j^0 \tag{1.28}$$

where $\check{\Gamma}$ is the vertical restriction of Γ.

Let K be a spacetime connection. Then [46], $\mathrm{d}\Omega = 0$ if and only if K is Galileian.

If K is a Galileian connection, then we obtain [49, 57]

$$\mathrm{d}t \wedge \Omega^3 \not\equiv 0 \qquad \mathrm{d}\Omega = 0 \qquad i(\gamma)\Omega = 0$$
$$L(\gamma)\Omega = 0 \qquad L(\gamma)\Lambda = 0 \qquad [\Lambda, \Lambda] = 0. \tag{1.29}$$

Hence, $(J_1\boldsymbol{E},\ \mathrm{d}t, \Omega)$ turns out to be a scaled cosymplectic manifold.

Now, let K be a Galileian connection. Then, we obtain the following results.

The cosymplectic 2-form Ω admits locally a *horizontal potential* $\Theta \in \mathrm{fib}(J_1\boldsymbol{E}, T^*\boldsymbol{E})$, defined up to a closed spacetime 1-form.

Next, let Θ be a horizontal potential of Ω and o an observer.

We define

- the *classical Lagrangian* as the horizontal phase 1-form $\mathcal{L}[\Theta] := \text{д}\lrcorner\Theta$;
- the *classical momentum* as the contact phase 1-form $\mathcal{P}[\Theta] := \theta\lrcorner\Theta$;
- the observed *classical Hamiltonian* as the horizontal phase 1-form $\mathcal{H}[\Theta, o] := -o\lrcorner\Theta$;
- the observed *classical momentum* as the vertical phase 1-form $\mathcal{P}[\Theta, o] := \nu[o]\lrcorner\Theta$;
- the observed *classical potential* as the phase 1-form $A[\Theta, o] := o^*\Theta$.

We obtain

$$\Theta = \mathcal{L}[\Theta] + \mathcal{P}[\Theta] = -\mathcal{H}[\Theta, o] + \mathcal{P}[\Theta, o]$$
$$\mathcal{L}[\Theta] = -\mathcal{H}[\Theta, o] + \text{д}\lrcorner\mathcal{P}[\Theta, o] \qquad \mathcal{P}[\Theta, o] = \nu[o]\lrcorner\mathcal{P}[\Theta]. \tag{1.30}$$

Moreover, we obtain $\mathrm{d}A[\Theta, o] = \Phi[K, o] := o^*\Omega$. Hence, $A[\Theta, o]$ turns out to be one of the potentials $A[K, o]$; thus, the choice of Θ fixes the gauge of $A[K, o]$.

Furthermore, the Euler–Lagrange equation associated with $\mathcal{L}[\Theta]$ turns out to be just the Newton law $\nabla[\gamma]j_1 s = 0$.

In a chart adapted to o we have the following coordinate expressions:

$$\Theta = -\tfrac{1}{2}G^0_{ij}x_0^i x_0^j d^0 + G^0_{ij}x_0^j d^i + A_\lambda d^\lambda \qquad A[\Theta, o] = A_\lambda d^\lambda$$
$$\mathcal{L}[\Theta] = \left(\tfrac{1}{2}G^0_{ij}x_0^i x_0^j + A_i x_0^i + A_0\right) d^0 \qquad \mathcal{P}[\Theta] = \left(G^0_{ij}x_0^j + A_i\right)\left(d^i - x_0^i d^0\right) \tag{1.31}$$
$$\mathcal{H}[\Theta, o] = \left(\tfrac{1}{2}G^0_{ij}x_0^i x_0^j - A_0\right) d^0 \qquad \mathcal{P}[\Theta, o] = \left(G^0_{ij}x_0^j + A_i\right)d^i.$$

1.8. Gravitational and electromagnetic fields

Now, we introduce the gravitational and electromagnetic fields.

The *gravitational* and *electromagnetic fields* are defined to be, respectively, a Galileian spacetime connection and a scaled closed 2-form of spacetime

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \underset{T\boldsymbol{E}}{\otimes} TT\boldsymbol{E} \tag{1.32}$$

and

$$f : \boldsymbol{E} \to \left(\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}\right) \otimes \Lambda^2 T^* \boldsymbol{E}. \tag{1.33}$$

With reference to a charge q, we define the *re-scaled electromagnetic field*

$$F = \frac{q}{\hbar} f : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E}. \tag{1.34}$$

As the third *postulate* of our classical model, we assume a gravitational field $K^\natural$ and an electromagnetic field F.

Then, it is convenient to introduce the *total connection*

$$K := K^\natural + K^e = K^\natural + (\mathrm{d}t \otimes \widehat{F} + \widehat{F} \otimes \mathrm{d}t) \tag{1.35}$$

where $\widehat{F} = G_0^{ih} F_{h\mu} u^0 \otimes \partial_i \otimes d^\mu$.

The total connection K turns out to be Galileian.

In the following we shall refer to the total connection K.

Clearly, the objects induced on the phase space by the total connection K split into their gravitational and electromagnetic components.

2. Quantum structure

We are concerned with quantum mechanics of a spinless charged particle on a curved spacetime fibred over absolute time and equipped with a given spacelike metric, gravitational field and electromagnetic field.

This section is devoted to our postulates on the quantum theory and their first consequences. We just assume a quantum bundle, a quantum metric and a quantum connection. All further objects are derived from them by a covariant procedure.

2.1. Quantum bundle

We start with the quantum bundle over spacetime.

As the first *postulate* of our quantum model, we assume the *quantum bundle* to be a one-dimensional complex bundle over spacetime $\pi : \boldsymbol{Q} \to \boldsymbol{E}$.

Each section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ is called a *quantum section*.

Let us consider the trivial bundle $\mathcal{L}(\boldsymbol{Q}) = \boldsymbol{E} \times (\mathbb{R}^+ \times U(1))$ over $\boldsymbol{E}$ of *complex linear automorphisms* $\lambda : \boldsymbol{Q} \to \boldsymbol{Q}$ over $\boldsymbol{E}$.

For each $s \geqslant r \geqslant 0$, we obtain the natural fibred action

$$\begin{aligned} \nu_{s,r} &: \mathcal{G}^s(\boldsymbol{E}) \underset{\boldsymbol{E}}{\rtimes} J_r\mathcal{L}(\boldsymbol{Q}) \to J_r\mathcal{L}(\boldsymbol{Q}) \\ &: (j_s\phi(e), j_r\lambda(e)) \mapsto j_r(\lambda \circ \phi)(e). \end{aligned}$$

For each $s \geqslant r \geqslant 0$, we define the *quantum covariance bundle*, of order (s, r), to be the bundle

$$\mathcal{W}^{(s,r)}(\boldsymbol{Q}) := \mathcal{G}^s(\boldsymbol{E}) \underset{\boldsymbol{E}}{\rtimes} J_r\mathcal{L}(\boldsymbol{Q}) \to \boldsymbol{E}. \tag{2.1}$$

The bundle $\mathcal{W}^{(s,r)}(\boldsymbol{Q})$ turns out a group bundle through the composition fibred morphism over $\boldsymbol{E}$

$$\mu^{(s,r)} : \mathcal{W}^{(s,r)}(\boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} \mathcal{W}^{(s,r)}(\boldsymbol{Q}) \to \mathcal{W}^{(s,r)}(\boldsymbol{Q}) :$$

$$\left(\left(j_s \overset{1}{\phi}(e), j_r \overset{1}{\lambda}(e)\right), \left(j_s \overset{2}{\phi}(e), j_r \overset{2}{\lambda}(e)\right)\right) \mapsto \left(\left(j_s(\overset{1}{\phi} \circ \overset{2}{\phi})\right)(e), j_r\left(\left(\overset{2}{\lambda} \circ \overset{1}{\phi}\right) \cdot \overset{1}{\lambda}\right)(e)\right).$$

For $s \geqslant r \geqslant 0$ and $s - k \geqslant r - h \geqslant 0$, we obtain a natural fibred group epimorphism $\mathcal{W}^{(s,r)}(\boldsymbol{Q}) \to \mathcal{W}^{(s-k,r-h)}(\boldsymbol{Q})$ over $\boldsymbol{E}$.

We denote the fibred chart of $\mathcal{W}^{(s,r)}(\boldsymbol{Q})$ induced by a spacetime chart (x^λ) by $\big(x^\lambda; \mathfrak{g}^0_0, \mathfrak{g}^i_\lambda, \ldots, \mathfrak{g}^i_{\lambda_1\ldots\lambda_s}; c, c_\lambda, c_{\lambda_1\ldots\lambda_r}\big)$.

For instance, we obtain the following coordinate expressions

$$\big(x^\lambda; c, c_\lambda\big) \circ \nu_{1,1} = \big(x^\lambda; c, c_\mu \mathfrak{g}^\mu_\lambda\big)$$
$$\big(x^\lambda; c, c_\lambda, c_{\lambda\mu}\big) \circ \nu_{2,2} = \big(x^\lambda; c, c_\mu \mathfrak{g}^\mu_\lambda, c_{\rho\sigma}\mathfrak{g}^\rho_\lambda \mathfrak{g}^\sigma_\mu + c_\nu \mathfrak{g}^\nu_{\lambda\mu}\big) .$$

In view of the quantum connection, we consider also the following bundle.

The *extended quantum bundle* is defined to be the pullback bundle

$$\pi^\uparrow : \boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E} \tag{2.2}$$

of the quantum bundle with respect to the map $J_1\boldsymbol{E} \to \boldsymbol{E}$.

Here, the extended base space $J_1\boldsymbol{E}$ plays the role of a space of observers.

2.2. *Quantum metric*

Next, we introduce the Hermitian metric on the quantum bundle. In view of the integration on the fibres of spacetime, this metric has values in the space of spacelike volume forms.

We define a *quantum metric* to be a fibred Hermitian metric with values in the complexified space of vertical volume forms

$$\mathrm{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{C} \otimes \Lambda^3 V^*\boldsymbol{E}. \tag{2.3}$$

The quantum metrics are the sections of a trivial principal bundle $\mathrm{Met}(\boldsymbol{Q}) \to \boldsymbol{E}$, whose structure group is $\mathbb{R}^+$.

Moreover, we have a natural fibred action $\mathcal{W}^{(1,0)}(\boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{Q}) \to \mathrm{Met}(\boldsymbol{Q})$.

As the second *postulate* of our quantum model, we assume a quantum metric h.

Clearly, $\mathrm{h}' := \mathrm{h}/\eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{*3} \otimes \mathbb{C}$ is a standard scaled Hermitian metric.

We define a *quantum basis* to be a section $\mathrm{b} \in \sec(\boldsymbol{E}, \mathbb{L}^{3/2} \otimes \boldsymbol{Q})$ such that $\mathrm{h}(\mathrm{b}, \mathrm{b}) = \eta$.

If b is a quantum basis, then an associated *quantum chart* is defined to be a complex linear fibred chart (x^λ, z), where (x^λ) is a spacetime chart and $z \in \mathrm{map}(\boldsymbol{Q}, \mathbb{C} \otimes \mathbb{L}^{*3/2})$ is the complex dual of b.

We shall refer to quantum bases and quantum charts.

Each quantum basis b yields the *real quantum basis* $(\mathrm{b}_1, \mathrm{b}_2) := (\mathrm{b}, \mathrm{i}\mathrm{b})$ and the *real quantum chart* $(w^1, w^2) = \big(\frac{1}{2}(z + \bar{z}), \frac{1}{2}(\bar{z} - z)\big)$.

If $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, then we write $\Psi = \psi\mathrm{b}$ and obtain $\mathrm{h}(\Phi, \Psi) = \bar{\phi}\psi\eta$.

Let us consider two quantum bases b and $\acute{\mathrm{b}}$ and the associated fibre coordinates z and $\acute{z}$. Then, we obtain $\acute{\mathrm{b}} = \exp(\mathrm{i}\vartheta)\mathrm{b}$ and $\acute{z} = \exp(-\mathrm{i}\vartheta)z$, with $\vartheta \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

Hence, the quantum bundle turns out to be a bundle associated with a principal bundle, whose structure group is $U(1)$.

The quantum metric h yields, by pullback, the *extended quantum metric*

$$\mathrm{h}^\uparrow : \boldsymbol{Q}^\uparrow \underset{J_1\boldsymbol{E}}{\times} \boldsymbol{Q}^\uparrow \to \mathbb{C} \otimes \Lambda^3 V^*\boldsymbol{E}.$$

2.3. *Quantum connection*

Eventually, we complete the quantum framework by introducing the quantum connection. We introduce this notion by three steps.

In our formulation we use some notions (for instance, universal connections) which are non-standard, hence deserve a clear analysis. Other notions are essentially standard, but our setting involves some delicate aspects which need an explicit mention.

2.3.1. Connections of the quantum bundle. First, we analyse the Hermitian connections of the quantum bundle.

Let us refer to a quantum basis $\mathfrak{b}$ and to an adapted quantum chart (x^λ, z). The complex linear connections Ψ of $\boldsymbol{Q}$ are of the type [46]

$$Ч = \chi[\mathfrak{b}] + ч[\mathfrak{b}] \otimes \mathbb{I} = d^\lambda \otimes \partial_\lambda + ч_\lambda z d^\lambda \otimes \mathfrak{b} \tag{2.4}$$

where $\chi[\mathfrak{b}]$ is the flat connection induced by $\mathfrak{b}$, $\mathbb{I} = z \otimes \mathfrak{b}$ is the Liouville vector field of $\boldsymbol{Q}$ and $ч[\mathfrak{b}] \equiv ч_\lambda d^\lambda \in \sec(\boldsymbol{E}, \mathbb{C} \otimes T^*\boldsymbol{E})$ is a complex spacetime form.

The complex linear connections of $\boldsymbol{Q}$ are the sections of a bundle $\mathrm{Con}(\boldsymbol{Q}) \to \boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields a local fibred isomorphism $\mathrm{Con}(\boldsymbol{Q}) \to \mathbb{C} \otimes T^*\boldsymbol{E}$ over $\boldsymbol{E}$.

A complex linear connection $Ч$ of $\boldsymbol{Q}$ is said to be *Hermitian* if $\nabla \mathfrak{h} = 0$, where ∇ is the covariant differential induced by $Ч$ and K.

Indeed, we are involved also with K because $\mathfrak{h}$ has values in $\mathbb{C} \otimes \Lambda^3 V^*\boldsymbol{E}$.

Let us refer to a quantum basis $\mathfrak{b}$ and to an adapted quantum chart (x^λ, z). The Hermitian connections $Ч$ of $\boldsymbol{Q}$ are of the type [46]

$$Ч = \chi[\mathfrak{b}] + \mathrm{i}A[\mathfrak{b}] \otimes \mathbb{I} = d^\lambda \otimes \partial_\lambda + \mathrm{i}A_\lambda d^\lambda \otimes \mathbb{I} \tag{2.5}$$

where $A[\mathfrak{b}] \equiv A_\lambda d^\lambda \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ is a spacetime form, called the *potential* of $Ч$.

Let us consider a Hermitian connection $Ч$ of $\boldsymbol{Q}$. Then, with reference to two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \exp(\mathrm{i}\vartheta)\mathfrak{b}$, we obtain the transition map $A[\acute{\mathfrak{b}}] = A[\mathfrak{b}] - \mathrm{d}\vartheta$.

Hence, with reference to adapted quantum charts (x^λ, z) and $(\acute{x}^\lambda, \acute{z})$, we obtain the transition maps

$$\acute{A}_i = (A_j - \partial_j\vartheta)\sigma_i^j \qquad \acute{A}_0 = (A_0 - \partial_0\vartheta)\sigma_0^0 + (A_j - \partial_j\vartheta)\sigma_0^j. \tag{2.6}$$

The Hermitian connections of $\boldsymbol{Q}$ are the sections of a bundle $\mathrm{Her}(\boldsymbol{Q}) \to \boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields locally a fibred isomorphism $\mathrm{Her}(\boldsymbol{Q}) \to T^*\boldsymbol{E}$ over $\boldsymbol{E}$.

Analogous notions and constructions hold for the connections of the extended quantum bundle.

2.3.2. Universal connections of the extended quantum bundle. Next, we introduce the notion of universal connection [6, 41, 46]. Let $\{\overset{o}{Ч}\} \equiv \{Ч[o]\}$ be a 'system' of complex linear connections of the quantum bundle parametrized by the observers $o \in \sec(\boldsymbol{E}, J_1\boldsymbol{E})$.

Then, there is a unique complex linear connection $Ч^\uparrow$ of the extended quantum bundle, called *universal*, such that $Ч[o] = o^*Ч^\uparrow$, for each o.

Now, let us consider a quantum basis $\mathfrak{b}$ and denote by $ч[\mathfrak{b}, o] \in \sec(\boldsymbol{E}, \mathbb{C} \otimes T^*\boldsymbol{E})$ the complex form associated with the connection $Ч[o]$. Then, there is a unique horizontal complex phase form $ч^\uparrow[\mathfrak{b}] = ч^\uparrow_\lambda d^\lambda \in \mathrm{fib}(J_1\boldsymbol{E}, \mathbb{C} \otimes T^*\boldsymbol{E})$, such that $ч[\mathfrak{b}, o] = ч^\uparrow[\mathfrak{b}] \circ o$.

Hence, the expression of $Ч^\uparrow$ is

$$\begin{aligned} Ч^\uparrow &= \chi^\uparrow[\mathfrak{b}] + ч^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow \\ &= d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + ч^\uparrow_\lambda d^\lambda \otimes \mathbb{I}^\uparrow \end{aligned} \tag{2.7}$$

where $\chi^\uparrow[\mathfrak{b}]$ is the flat Hermitian connection induced by $\mathfrak{b}$ and $\mathbb{I}^\uparrow = z \otimes \mathfrak{b}$ is the Liouville vector field of $\boldsymbol{Q}^\uparrow$.

Conversely, each complex linear connection $\Psi^\uparrow$ of $\boldsymbol{Q}^\uparrow$ of the above type yields a system of complex linear connections of $\boldsymbol{Q}$, whose universal connection is $\Psi^\uparrow$.

If $\{\Psi[o]\}$ is a system of complex linear connections of the quantum bundle, then we obtain, for each observer o,

$$R[\Psi[o]] = o^* R[\Psi^\uparrow]. \tag{2.8}$$

The universal connections of $\boldsymbol{Q}^\uparrow$ are the sections of a bundle $\mathrm{Uni}(\boldsymbol{Q}^\uparrow) \to J_1\boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields a local fibred isomorphism $\mathrm{Uni}(\boldsymbol{Q}^\uparrow) \to \mathbb{C} \otimes (J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E})$ over $J_1\boldsymbol{E}$.

A complex linear connection $\Psi^\uparrow$ of $\boldsymbol{Q}^\uparrow$ is said to be *Hermitian* if $\nabla \mathrm{h}^\uparrow = 0$, where ∇ is the covariant differential induced by $\Psi^\uparrow$ and K.

The Hermitian connections of $\boldsymbol{Q}^\uparrow$ are the sections of a bundle $\mathrm{Her}(\boldsymbol{Q}^\uparrow) \to J_1\boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields a local fibred isomorphism $\mathrm{Her}(\boldsymbol{Q}^\uparrow) \to T^*J_1\boldsymbol{E}$ over $J_1\boldsymbol{E}$.

Let $\{\Psi[o]\}$ be a system of complex linear connections of $\boldsymbol{Q}$ and $\Psi^\uparrow$ its universal connection. Then, the connections $\{\Psi[o]\}$ are Hermitian if and only if $\Psi^\uparrow$ is Hermitian.

Let us refer to a quantum basis $\mathfrak{b}$ and to an adapted quantum chart (x^λ, z). Then, the Hermitian universal connections $\Psi^\uparrow$ of $\boldsymbol{Q}^\uparrow$ are of the type [46]

$$\begin{aligned} \Psi^\uparrow &= \chi^\uparrow[\mathfrak{b}] + \mathrm{i}A^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow \\ &= d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathrm{i}A^\uparrow_\lambda d^\lambda \otimes \mathbb{I}^\uparrow \end{aligned} \tag{2.9}$$

where $A^\uparrow[\mathfrak{b}] \equiv A^\uparrow_\lambda d^\lambda \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ is a horizontal phase form.

Let us consider a universal Hermitian connection $\Psi^\uparrow$ of $\boldsymbol{Q}^\uparrow$. Then, two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \exp(\mathrm{i}\vartheta)\mathfrak{b}$ yield the transition map $A^\uparrow[\acute{\mathfrak{b}}] = A^\uparrow[\mathfrak{b}] - \mathrm{d}\vartheta$. Hence, with reference to adapted quantum charts (x^λ, z) and $(\acute{x}^\lambda, \acute{z})$, we obtain the transition maps $\acute{A}^\uparrow_i = (A^\uparrow_j - \partial_j\vartheta)\sigma^j_i$ and $\acute{A}^\uparrow_0 = (A^\uparrow_0 - \partial_0\vartheta)\sigma^0_0 + (A^\uparrow_j - \partial_j\vartheta)\sigma^j_0$.

The universal Hermitian connections of $\boldsymbol{Q}^\uparrow$ are the sections of the bundle $\mathrm{Uni}(\boldsymbol{Q}^\uparrow) \cap \mathrm{Her}(\boldsymbol{Q}^\uparrow) \to J_1\boldsymbol{E}$. Moreover, each quantum basis $\mathfrak{b}$ yields a local fibred isomorphism $\mathrm{Uni}(\boldsymbol{Q}^\uparrow) \cap \mathrm{Her}(\boldsymbol{Q}^\uparrow) \to J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E}$ over $J_1\boldsymbol{E}$.

2.3.3. Quantum connections. Eventually, we introduce the notion of quantum connection.

Now, we take into account the vertical metric G the total connection K and the induced cosymplectic 2-form Ω.

We define [46, 49] a *quantum connection* to be a connection $\Psi^\uparrow$ of the extended quantum bundle which is universal, Hermitian and whose curvature is

$$R[\Psi^\uparrow] = -2\mathrm{i}\Omega \otimes \mathbb{I}^\uparrow. \tag{2.10}$$

We recall that Ω includes $m/\hbar$ through the re-scaled metric G and that it includes both the gravitational and electromagnetic fields through the total connection K.

Let $\{\Psi[o]\}$ be a system of Hermitian connections and $\Psi^\uparrow$ its universal connection. Then,

$$R[\Psi^\uparrow] = -2\mathrm{i}\Omega \otimes \mathbb{I}^\uparrow \quad \Leftrightarrow \quad R[\Psi[o]] = -2\mathrm{i}o^*\Omega \otimes \mathbb{I} \tag{2.11}$$

for all observers o.

A quantum connection $\Psi^\uparrow$ exists locally. Indeed, with reference to a quantum basis $\mathfrak{b}$, its local expression is of the type

$$\begin{aligned} \Psi^\uparrow &= \chi^\uparrow[\mathfrak{b}] + \mathrm{i}A^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow \\ &= d^\lambda \otimes \partial_\lambda + d^i_0 \otimes \partial^0_i + \mathrm{i}A^\uparrow_\lambda d^\lambda \otimes \mathbb{I}^\uparrow \end{aligned} \tag{2.12}$$

with $\mathrm{d}A^\uparrow[\mathfrak{b}] = \Omega$.

Hence, for each observer o, if we set $A[\mathrm{b}, o] := o^* A^\uparrow[\mathrm{b}] = A_\lambda d^\lambda$, then we obtain

$$A^\uparrow[\mathrm{b}] \equiv A^\uparrow_\lambda d^\lambda = -\left(\tfrac{1}{2} G^0_{ij} x^i_0 x^j_0 - A_0\right) d^0 + \left(G^0_{ij} x^j_0 + A_i\right) d^i \tag{2.13}$$

with $\mathrm{d}A[\mathrm{b}, o] = \Phi[o]$.

If $\text{Ч}^\uparrow$ is a quantum connection, then, for each quantum basis b, $A^\uparrow[\mathrm{b}]$ turns out to be a horizontal potential of Ω, whose gauge is determined by b. Moreover, for each observer o, $A[\mathrm{b}, o]$ turns out to be a potential of K.

Let us consider a quantum connection $\text{Ч}^\uparrow$, two quantum bases $\mathrm{b}, \acute{\mathrm{b}} = \exp(\mathrm{i}\vartheta)\mathrm{b}$ and two observers $o, \acute{o} = o + v$. Then, we obtain the transition map

$$A[\acute{\mathrm{b}}, \acute{o}] = A[\mathrm{b}, o] - \tfrac{1}{2} G(v, v) + v[o] \lrcorner G^\flat(v) - \mathrm{d}\theta. \tag{2.14}$$

Thus, each connection of the system determines all others by the above formula.

The transition maps of (2.14) well define a fibred group action over $\boldsymbol{E}$

$$\mathcal{T}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} (J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathrm{Her}(\boldsymbol{Q})) \quad \to \quad J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathrm{Her}(\boldsymbol{Q}).$$

The quantum connections are the sections of a bundle $\mathrm{Con}(\boldsymbol{Q}^\uparrow)$ *over spacetime*. We have a natural fibred group action $\mathcal{W}^{(1,1)}(\boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} \mathrm{Con}(\boldsymbol{Q}^\uparrow) \to \mathrm{Con}(\boldsymbol{Q}^\uparrow)$.

The condition $R[\text{Ч}^\uparrow] = -2\mathrm{i}\Omega \otimes \mathbb{I}^\uparrow$ has reduced the base space of the bundle of universal Hermitian connections from the phase space to spacetime.

For each quantum basis b and observer o, we have the fibred isomorphism $\mathrm{Con}(\boldsymbol{Q}^\uparrow) \to T^*\boldsymbol{E} : \text{Ч}^\uparrow \mapsto A[\mathrm{b}, o]$.

Thus, given a quantum connection, the choice of a quantum basis b turns out to be another way to control the gauge of classical potentials.

Let $\text{Ч}^\uparrow$ be a quantum connection and b a quantum basis. Then, there is a unique observer $o[\mathrm{b}]$, such that the vertical restriction $\overset{\vee}{A}[\mathrm{b}, o[\mathrm{b}]]$ vanishes. Indeed, if o is any observer, then we obtain $o[\mathrm{b}] = o - G^\sharp(A[\mathrm{b}, o])$.

Let $\text{Ч}^\uparrow$ be a quantum connection and Ψ a quantum section. Then, in the domain where $\Psi \neq 0$, we obtain the distinguished observer

$$o[\Psi] := o[\mathrm{b}] \qquad \text{where} \quad \mathrm{b} := \Psi / \sqrt{h'(\Psi, \Psi)}. \tag{2.15}$$

A quantum connection $\text{Ч}^\uparrow$ exists globally if and only if the cohomology class of Ω is integer [99].

As the third *postulate* of our quantum model, we suppose that the quantum bundle admits quantum connections and assume a quantum connection $\text{Ч}^\uparrow$.

The equality $\mathrm{d}\Omega = 0$, which is a consequence of our assumption that K be Galileian, turns out to be an integrability condition for the equation $R[\text{Ч}^\uparrow] = -2\mathrm{i}\Omega \otimes \mathbb{I}^\uparrow$, by virtue of the Bianchi identity for $R[\text{Ч}^\uparrow]$.

On the other hand, if we had just assumed that K is a metric spacetime connection, then the assumption of a quantum connection would imply that K be Galileian.

2.3.4. Quantum differentials. The quantum connection $\text{Ч}^\uparrow$, the Galileian connection K and the spacelike metric G yield distinguished differential operators.

Given $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ and $o \in \sec(\boldsymbol{E}, J_1\boldsymbol{E})$, we obtain the following objects:

- the *quantum differential* related to $\text{Ч}^\uparrow$

$$\nabla\Psi = (\nabla_\lambda \psi) d^\lambda \otimes \mathrm{b} \qquad \text{with} \quad \nabla_\lambda \psi := (\partial_\lambda - \mathrm{i}A^\uparrow_\lambda)\psi; \tag{2.16}$$

- the *observed quantum differential* related to $\text{Ч}^{\uparrow}$ and o
$$\overset{o}{\nabla}\Psi = \left(\overset{o}{\nabla}_{\lambda}\psi\right) d^{\lambda} \otimes \mathsf{b} \qquad \text{with} \quad \overset{o}{\nabla}_{\lambda}\psi := (\partial_{\lambda} - \mathrm{i}A_{\lambda})\psi; \tag{2.17}$$
- the *timelike quantum differential* related to $\text{Ч}^{\uparrow}$
$$\widehat{\nabla}\Psi := \text{д}\lrcorner\nabla\Psi = \left(\partial_0 + x_0^j\partial_j - \mathrm{i}\mathcal{L}_0[A^{\uparrow}[\mathsf{b}]]\right)\psi u^0 \otimes \mathsf{b}; \tag{2.18}$$
- the *spacelike quantum differential* related to $\text{Ч}^{\uparrow}$
$$\overset{\vee}{\nabla}\Psi = (\partial_j - \mathrm{i}\mathcal{P}_j[A^{\uparrow}[\mathsf{b}]])\psi\check{d}^j \otimes \mathsf{b}; \tag{2.19}$$
- the *observed second quantum differential* related to $\text{Ч}^{\uparrow}$, K and o
$$\overset{o}{\nabla}\overset{o}{\nabla}\Psi = \overset{o}{\nabla}_{\lambda\mu}\psi d^{\lambda} \otimes d^{\mu} \otimes \mathsf{b} \tag{2.20}$$
with $\overset{o}{\nabla}_{\lambda\mu}\psi = \big(\overset{o}{\nabla}_{\lambda}\overset{o}{\nabla}_{\mu} + K_{\lambda}{}^{j}{}_{\mu}\overset{o}{\nabla}_{j}\big)\psi$;
- the *observed quantum Laplacian* related to G, $\text{Ч}^{\uparrow}$, K and o
$$\overset{o}{\Delta}\Psi := \langle\bar{G}, \overset{o}{\nabla}\overset{o}{\nabla}\Psi\rangle = \left(\overset{o}{\Delta}_0\psi\right) u^0 \otimes \mathsf{b} \tag{2.21}$$
with $\overset{o}{\Delta}_0\psi = G_0^{hk}\big(\overset{o}{\nabla}_h\overset{o}{\nabla}_k + \varkappa_h{}^{j}{}_{k}\overset{o}{\nabla}_j\big)\psi$, where we refer to a quantum basis b, to an observer o and to an adapted chart (x^{λ}, z).

Analogously, we obtain the quantum differentials and quantum Laplacians of sections of the real dual quantum bundle $\boldsymbol{Q}^* \to \boldsymbol{E}$.

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we have $(\mathrm{re}\,\mathsf{h})^{\flat}(\overset{o}{\Delta}\Psi) = \overset{o}{\Delta}((\mathrm{re}\,\mathsf{h})^{\flat}\Psi)$.

Later, we shall be involved with the following technical result.

Given two observers o and $\acute{o} = o + v$, we obtain
$$\begin{aligned}
\overset{\acute{o}}{\nabla} &= \overset{o}{\nabla} + \mathrm{i}\tfrac{1}{2}G(v, v) - \mathrm{i}\nu^*[o](G^{\flat}(v)) \\
\acute{o}\lrcorner\overset{\acute{o}}{\nabla} &= o\lrcorner\overset{o}{\nabla} - \mathrm{i}\tfrac{1}{2}G(v, v) + v\lrcorner\overset{o}{\nabla} \\
\overset{\acute{o}}{\Delta} &= \overset{o}{\Delta} - 2\mathrm{i}v\lrcorner\overset{o}{\nabla} - \mathrm{i}\left(C_1^1\nabla v\right) - G(v, v).
\end{aligned} \tag{2.22}$$

3. Quantum dynamics

This section is devoted to the derivation of the Schrödinger operator and the related matter from the quantum metric and the quantum connection.

We follow two approaches via covariant differentials and a covariant Lagrangian.

In both cases, we use a projectability criterion as a heuristic method. Namely, we start from objects living on the classical phase space, because this is the seat of the quantum connection; then, we look for the combinations of the above objects which project on spacetime. This procedure means that we get rid of observers. Indeed, this turns out to be an implementation of the principle of general relativity.

3.1. Differential approach

The quantum connection and the method of projectability yield good candidates for the quantum momentum, the Schrödinger operator and the probability current.

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we have the two distinguished maps living on the classical *phase space*:
$$\begin{aligned}
\text{д} \otimes \Psi &= \psi u^0 \otimes \left(\partial_0 + x_0^i\partial_i\right) \otimes \mathsf{b} \\
G^{\sharp}(\nabla\Psi) &= \left(G_0^{ij}\partial_j - \mathrm{i}\left(x_0^i + A_0^i\right)\right)\psi u^0 \otimes \partial_i \otimes \mathsf{b}.
\end{aligned}$$

Then, we obtain the section on *spacetime*

$$\begin{aligned} \mathrm{P}[\Psi] &:= д \otimes \Psi - \mathrm{i} G^{\sharp}(\nabla\Psi) \\ &= u^0 \otimes \left(\psi\partial_0 - \mathrm{i} G_0^{ij}(\partial_j - \mathrm{i} A_j)\psi\partial_i\right) \otimes \mathrm{b}. \end{aligned} \tag{3.1}$$

Thus, we obtain, in a covariant way, a global, gauge-independent and observer-independent first-order operator $\mathrm{P} : J_1\boldsymbol{Q} \to \mathbb{T}^* \otimes (T\boldsymbol{E} \underset{\boldsymbol{E}}{\otimes} \boldsymbol{Q})$, called *quantum momentum*.

If $\Psi \neq 0$, then we obtain the section

$$\begin{aligned} W[\Psi] &:= \frac{\mathrm{P}[\Psi]}{\Psi} = \frac{\mathrm{h}(\Psi, \mathrm{P}[\Psi])}{\mathrm{h}(\Psi, \Psi)} = o[\Psi] + \frac{G^{\sharp}(\overset{\vee}{\overset{o}{\nabla}}\Psi)}{\Psi} \\ &= u^0 \otimes \left(\partial_0 - A_0^i\partial_i\right) - \mathrm{i}\frac{G_0^{ij}\partial_j\psi}{\psi}\partial_i. \end{aligned}$$

The real component

$$\mathrm{re}\, W[\Psi] = u^0 \otimes \left(\partial_0 - A_0^i\partial_i\right) + \mathrm{i}\frac{1}{2}G_0^{ij}\frac{\psi\partial_j\bar{\psi} - \bar{\psi}\partial_j\psi}{\bar{\psi}\psi}u^0 \otimes \partial_i$$

turns out to be a distinguished observer associated with Ψ (see also (2.15)).

For each quantum section Ψ, we have the two distinguished maps living on the classical *phase space*:

$$\begin{aligned} д\lrcorner\nabla\Psi &= \left(\partial_0\psi + x_0^h\partial_h\psi - \mathrm{i}\mathcal{L}_0[A^{\uparrow}[\mathrm{b}]]\right)\psi u^0 \otimes \mathrm{b} \\ \delta\mathrm{P}[\Psi] &= \left(\partial_0 - 2\mathrm{i}A_0 + \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \mathrm{i}\overset{o}{\Delta} - x_0^h\partial_h + \mathrm{i}\mathcal{L}_0[A^{\uparrow}[\mathrm{b}]]\right)\psi \otimes u^0 \otimes \mathrm{b} \end{aligned}$$

where δ denotes the quantum codifferential. Then, we obtain the section on *spacetime*

$$\begin{aligned} \mathrm{S}[\Psi] &:= \tfrac{1}{2}(д\lrcorner\nabla\Psi + \delta\mathrm{P}[\Psi]) \\ &= \left(\overset{o}{\nabla}_0 + \tfrac{1}{2}(\mathrm{div}_\eta\, o)_0 - \mathrm{i}\tfrac{1}{2}\overset{o}{\Delta}_0\right)\psi u^0 \otimes \mathrm{b} \end{aligned} \tag{3.2}$$

where $\mathrm{div}_\eta\, o = \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}u^0$.

Thus, we obtain, in a covariant way, a global, gauge-independent and observer-independent second-order operator $\mathrm{S} : J_2\boldsymbol{Q} \to \mathbb{T}^* \otimes \boldsymbol{Q}$, called the *Schrödinger operator*.

Given any observer o, we can write $\mathrm{S}[\Psi] = (\mathrm{re}\,\mathrm{h})^{\sharp}\big(\overset{o}{\delta} - \frac{1}{2}\mathrm{i}\overset{o}{\Delta}\big)(\mathrm{re}\,\mathrm{h})^{\flat}(\Psi)$ where $\overset{o}{\delta}$ denotes the Lie derivative with respect to the quantum prolongation of the observer o. Clearly, $\overset{o}{\delta}$ and $\overset{o}{\Delta}$ depend on o, but the above combination turns out to be observer independent.

Given any observer o, the Schrödinger operator can be written as

$$\mathrm{S}[\Psi] = \tfrac{1}{2}(д\lrcorner\nabla\Psi + \delta\mathrm{P}[\Psi]) \circ o \tag{3.3}$$

where $(д\lrcorner\nabla\Psi) \circ o = o\lrcorner\overset{o}{\nabla}\Psi$ and $(\delta\mathrm{P}[\Psi]) \circ o = -\mathrm{i}\overset{o}{\Delta}\Psi + o\lrcorner\overset{o}{\nabla}\Psi + C_1^1(\nabla o)\Psi$.

Clearly, $(д\lrcorner\nabla\Psi) \circ o$ and $(\delta\mathrm{P}[\Psi]) \circ o$ depend on o, but the above combination turns out to be observer independent.

We recall that the observed quantum Laplacian includes $m/\hbar$ through the re-scaled metric G. Moreover, we recall that the quantum potential $A[\mathrm{b}, o]$ includes both the gravitational and electromagnetic potentials.

Clearly, in the flat case, S reduces to the standard Schrödinger operator.

The scalar curvature of G yields, in a covariant way, the timescale $r : \boldsymbol{E} \to \bar{\mathbb{T}}^*$. Hence, for each $k \in \mathbb{R}$, we obtain the extended map

$$\mathrm{S}_{(k)}[\Psi] := \mathrm{S}[\Psi] - \mathrm{i}\tfrac{1}{2} k r \otimes \Psi. \tag{3.4}$$

Thus, we obtain, in a covariant way, a global, gauge-independent and observer-independent second-order operator $\mathrm{S}_{(k)} : J_2\boldsymbol{Q} \to \mathbb{T}^* \otimes \boldsymbol{Q}$, called the *extended Schrödinger operator*.

We have considered an imaginary coefficient $-\mathrm{i}\frac{1}{2}k$ in order to obtain later a Hermitian connection on the quantum Hilbert bundle from the extended Schrödinger operator. There is no way of determining k by means of covariance arguments.

We stress that S involves only the first derivatives of the metric, while $\mathrm{S}_{(k)}$ involves the second derivatives of the metric. The above additional term arises also in the Feynman integral approach to the Schrödinger operator. It seems that in this context the coefficient k could be determined. However, there is no definite agreement on its value. On the other hand, if the scalar curvature is constant, then the additional term produces an overall shift in the energy spectrum, which can hardly be detected.

Clearly, in the flat case the additional term $-\mathrm{i}\frac{1}{2}kr \otimes \Psi$ vanishes.

For each quantum section Ψ, we obtain the scaled vector field

$$\begin{aligned}\mathrm{j}[\Psi] &:= \tfrac{1}{2}(\mathrm{h}'(\Psi, \mathrm{P}[\Psi]) + \mathrm{h}'(\mathrm{P}[\Psi], \Psi)) \\ &= (\bar{\psi}\psi) u^0 \otimes \partial_0 + G_0^{hk}\big(\mathrm{i}\tfrac{1}{2}(\psi \partial_h \bar{\psi} - \bar{\psi}\partial_h \psi) - A_h \bar{\psi}\psi\big) u^0 \otimes \partial_k. \end{aligned} \tag{3.5}$$

Thus, we obtain, in a covariant way, a global, gauge-independent and observer-independent first-order operator $\mathrm{j} : J_1\boldsymbol{Q} \to (\mathbb{T}^* \otimes \mathbb{L}^{*3}) \otimes T\boldsymbol{E}$, called the *probability current*.

For each quantum section Ψ and for each $k \in \mathbb{R}$, we obtain

$$\delta \mathrm{j}[\Psi] = \mathrm{h}'\big(\Psi, \mathrm{S}_{(k)}[\Psi]\big) + \mathrm{h}'\big(\mathrm{S}_{(k)}[\Psi], \Psi\big), \tag{3.6}$$

where δ denotes the codifferential of $\mathrm{j}[\Psi]$.

Hence, the quantum probability current is conserved along the solutions of the extended Schrödinger equation.

3.2. Lagrangian approach

The quantum connection and the method of projectability yield good candidates for the quantum Lagrangian, hence for the derived quantum momentum, the Schrödinger operator and probability current.

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we have the two distinguished maps living on the classical *phase space*:

$$\begin{aligned}\widehat{\mathrm{L}}[\Psi] &:= \mathrm{d}t \wedge (\mathrm{h}(\widehat{\nabla}\Psi, \Psi) - \mathrm{h}(\Psi, \widehat{\nabla}\Psi)) \\ &= \Big((\psi\partial_0\bar{\psi} - \bar{\psi}\partial_0\psi) + 2\mathrm{i}A_0\bar{\psi}\psi - x_0^i(\bar{\psi}\partial_i\psi - \psi\partial_i\bar{\psi}) \\ &\quad + 2\mathrm{i}\left(\tfrac{1}{2}G_{ij}^0 x_0^i x_0^j + x_0^i A_i\right)\bar{\psi}\psi\Big)\upsilon^0\end{aligned}$$

$$\begin{aligned}\check{\mathrm{L}}[\Psi] &:= \mathrm{i}\,\mathrm{d}t \wedge ((\bar{G} \otimes \mathrm{h})(\check{\nabla}\Psi, \check{\nabla}\Psi)) \\ &= \Big(\mathrm{i}G_0^{ij}(\partial_i\bar{\psi}\partial_j\psi + A_iA_j\bar{\psi}\psi) - A_0^i(\bar{\psi}\partial_i\psi - \psi\partial_i\bar{\psi}) - x_0^i(\bar{\psi}\partial_i\psi - \psi\partial_i\bar{\psi}) \\ &\quad + \mathrm{i}\left(G_{ij}^0 x_0^i x_0^j + 2x_0^i A_i\right)\bar{\psi}\psi\Big)\upsilon^0.\end{aligned}$$

Then, we obtain the section on *spacetime*

$$\mathsf{L}[\Psi] := \tfrac{1}{4}\,\mathrm{d}t \wedge (\mathrm{h}(\Psi, \widehat{\nabla}\Psi) - \mathrm{h}(\widehat{\nabla}\Psi, \Psi) - \mathrm{i}(\bar{G} \otimes \mathrm{h})(\overset{\vee}{\nabla}\Psi, \overset{\vee}{\nabla}\Psi))$$
$$= \tfrac{1}{4}\big((\psi\partial_0\bar{\psi} - \bar{\psi}\partial_0\psi) + 2\mathrm{i}A_0\bar{\psi}\psi - \mathrm{i}G_0^{ij}(\partial_i\bar{\psi}\partial_j\psi + A_iA_j\bar{\psi}\psi)$$
$$+ A_0^i(\bar{\psi}\partial_i\psi - \psi\partial_i\bar{\psi})\big)\upsilon^0. \tag{3.7}$$

Thus, we obtain, in a covariant way, a global, gauge-independent and observer-independent first-order imaginary operator $\mathsf{L} : J_1\boldsymbol{Q} \to \mathrm{i}\Lambda^4 T\boldsymbol{E}$, called the *quantum Lagrangian*.

The normalizing coefficient $\frac{1}{4}$ has been chosen in view of further developments.

The scalar curvature of G yields, in a covariant way, the timescale $r : \boldsymbol{E} \to \bar{\mathbb{T}}^*$. Hence, for each $k \in \mathbb{R}$ we obtain the extended map

$$\mathsf{L}_{(k)}[\Psi] := \mathsf{L}[\Psi] + \mathrm{i}kr \wedge \mathrm{h}(\Psi, \Psi). \tag{3.8}$$

Thus, we obtain, in a covariant way, a global, gauge-independent and observer-independent first-order operator, called the *extended quantum Lagrangian*, $\mathsf{L}_{(k)} : J_1\boldsymbol{Q} \to \mathrm{i}\Lambda^4 T^*\boldsymbol{E}$.

For each $k \in \mathbb{R}$, the momentum associated with the extended quantum Lagrangian turns out to be given by $(\mathrm{re}\,\mathrm{h})^\sharp(V_0\mathsf{L}) = (\mathrm{re}\,\mathrm{h})^\sharp(V_0\mathsf{L}_{(k)}) = \frac{1}{2}\mathsf{P}$.

The Euler–Lagrange operator associated with the extended quantum Lagrangian turns out to be given by $\mathsf{S}_{(k)} = (\mathrm{re}\,\mathrm{h})^\sharp(\mathsf{E}[\mathsf{L}_{(k)}])$.

The Poincaré–Cartan form associated with the extended quantum Lagrangian turns out to be the map

$$\Theta\left[\mathsf{L}_{(k)}\right] := \mathsf{L}_{(k)} + \theta^*\left(V_0\mathsf{L}_{(k)}\right)$$
$$= \left(\tfrac{1}{8}G_0^{hk}(z_h\,z_k - \bar{z}_h\,\bar{z}_k) + \left(A_0 - \tfrac{1}{2}A_0^h\,A_h + \tfrac{1}{2}kr_0\right)z\bar{z}\right) \wedge \upsilon^0 + \tfrac{1}{2}(z\,\mathrm{d}\bar{z} - \bar{z}\,\mathrm{d}z) \wedge \upsilon_0^0$$
$$- \tfrac{1}{2}G_0^{hk}(\mathrm{i}(z_h\,\mathrm{d}\bar{z} + \bar{z}_h\,\mathrm{d}z) + A_h(z\,\mathrm{d}\bar{z} - \bar{z}\,\mathrm{d}z)) \wedge \upsilon_k^0. \tag{3.9}$$

The extended quantum Lagrangian is equivariant with respect to the action of the group $U(1)$. Hence, the vertical vector field $\mathrm{i}\mathbb{I} = \mathrm{i}z\mathsf{b}$ is an infinitesimal symmetry of the extended quantum Lagrangian.

For each $k \in \mathbb{R}$, the current

$$i(\mathrm{i}\mathbb{I})\Theta\left[\mathsf{L}_{(k)}\right] = i(\mathrm{i}\mathbb{I})\Theta[\mathsf{L}]$$
$$= (z\bar{z})\upsilon_0^0 + G_0^{hk}\left(\mathrm{i}\tfrac{1}{2}(z\bar{z}_h - \bar{z}z_h) - A_h z\bar{z}\right)\upsilon_k^0 \tag{3.10}$$

is conserved along the solutions of the extended Schrödinger equation. Actually, we obtain $i(\mathrm{i}\mathbb{I})\Theta[\mathsf{L}] = i(\mathrm{j})\upsilon$.

3.3. Uniqueness by covariance

We have found a Schrödinger operator and a quantum Lagrangian by a covariant geometric procedure and have shown that the Schrödinger operator can be derived from the quantum Lagrangian. Then, the following natural questions arise:

- Are there other possible covariant Schrödinger operators?
- Are there other possible quantum Lagrangians?
- Are all covariant Schrödinger operators derived from a quantum Lagrangian?

We answer these questions by proving that, under reasonable weak conditions concerning the order of derivatives, our Schrödinger operator and our quantum Lagrangian are essentially unique, provided they are covariant with respect to the quantum covariance group and to the change of timescale.

3.3.1. Covariant operators. First of all, we recall the notion of covariance and formulate our general problem.

Let us consider two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p' : \boldsymbol{F}' \to \boldsymbol{B}$ equipped with fibred actions $\boldsymbol{G} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{F}$ and $\boldsymbol{G} \underset{\boldsymbol{B}}{\times} \boldsymbol{F}' \to \boldsymbol{F}'$ of a group bundle $q : \boldsymbol{G} \to \boldsymbol{B}$.

Moreover, let us consider a subsheaf $\widetilde{\sec}(\boldsymbol{B}, \boldsymbol{F}) \subset \sec(\boldsymbol{B}, \boldsymbol{F})$, which is invariant with respect to the action of $\boldsymbol{G}$.

We define an *r-order covariant fibred morphism* to be a fibred morphism $\phi : J_r\boldsymbol{F} \to \boldsymbol{F}'$, which is equivariant with respect to the fibred action of the group bundle $\boldsymbol{G} \to \boldsymbol{B}$. Moreover, we define an *r-order covariant operator* to be a sheaf morphism $\mathfrak{O} : \widetilde{\sec}(\boldsymbol{B}, \boldsymbol{F}) \to \sec(\boldsymbol{B}, \boldsymbol{F}')$, which factorizes as $\mathfrak{O} = \phi \circ j_k$, where ϕ is an r-order covariant fibred morphism.

In the previous sections, we have assumed a spacelike metric, a gravitational field, an electromagnetic field, a quantum metric and a quantum connection. Now, in order to prove the above uniqueness theorems, we need to consider the above objects as variables.

So, here we assume the spacetime fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ and the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ and consider:

- the bundle $\mathrm{Met}(\boldsymbol{E})$ of re-scaled spacelike metrics G
- the bundle $\mathrm{Con}(\boldsymbol{E})$ of spacetime connections K
- the bundle $\mathrm{Met}(\boldsymbol{Q})$ of quantum metrics h
- the bundle $\mathrm{Uni}(\boldsymbol{Q}^{\uparrow})$ of universal connections $\mathrm{Ч}^{\uparrow}$.

We could define the covariant operators dealing directly with the universal connection $\mathrm{Ч}^{\uparrow}$, but we would meet cumbersome constructions, because this connection lives on the classical phase space. Thus, it is more convenient to deal with the system of connections $\{\overset{o}{\mathrm{Ч}}\}$ and their transition maps, as they live on spacetime.

Therefore, we introduce the following preliminary notions.

We define the sheaf of *observed fundamental fields* to be the sheaf

$$\overset{o}{\mathrm{fields}}(\boldsymbol{Q}) \subset \sec(\boldsymbol{E}, J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathrm{Con}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \mathrm{Met}(\boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} \mathrm{Con}(\boldsymbol{Q}))$$

which is constituted by all $(o, G, K, \mathrm{h}, \mathrm{Ч})$ fulfilling the *fundamental identities*

$$\nabla[K]G = 0 \qquad \mathrm{d}\Omega[G, K] = 0 \qquad \nabla[\mathrm{Ч}]\mathrm{h} = 0 \qquad R[\mathrm{Ч}] = -2\mathrm{i}o^*\Omega[G, K] \otimes \mathbb{I}.$$

Moreover, for each integer $0 \leqslant r$, we set

$$\overset{o}{\mathrm{fields}}_r(\boldsymbol{Q}) := \{(j_{r-1}o, j_rG, j_{r-1}K, j_r\mathrm{h}, j_{r-1}\mathrm{Ч}) \mid (o, G, K, \mathrm{h}, \mathrm{Ч}) \in \overset{o}{\mathrm{fields}}(\boldsymbol{Q})\}$$

and denote the corresponding jet prolongation by $j_r : \overset{o}{\mathrm{fields}}(\boldsymbol{Q}) \to \overset{o}{\mathrm{fields}}_r(\boldsymbol{Q})$.

Now, let us consider a fibred manifold $\boldsymbol{F} \to \boldsymbol{E}$, with a fibred action of the group bundle $\mathcal{W}^{(1,0)}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$ and a sheaf morphism $\mathfrak{O} : \overset{o}{\mathrm{fields}}(\boldsymbol{Q}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{F})$.

We say that $\mathfrak{O}$ is

- of *order r*, if it factorizes as $\mathfrak{O} = \mathfrak{O}_r \circ j_r$ through the r-jet prolongation of sections;
- *covariant* if it is invariant with respect to the fibred action of the transition group bundle $\mathcal{T}(\boldsymbol{E})$ and, for each observer o, the induced sheaf morphism $\overset{o}{\mathfrak{O}} := \mathfrak{O}(o, \cdot)$ is covariant with respect to the fibred action of the group bundles $\mathcal{W}^{(s,r)}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$.

Hence, if $\mathfrak{O}$ is covariant and $(G, K, \mathrm{h}, \mathrm{Ч}^{\uparrow})$ are the postulated fundamental fields, then the sheaf morphism $\mathrm{O} := \mathfrak{O}(o, G, K, \mathrm{h}, \overset{o}{\mathrm{Ч}}, \cdot) : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{F})$ does not depend on the choice of the observer o.

3.3.2. Schrödinger operator. We prove that all second-order Schrödinger operators are a complex linear combination of the Schrödinger operators exhibited in (3.2) and (3.4).

We define a *Schrödinger operator* to be a sheaf morphism

$$\mathfrak{O} : \overset{o}{\text{fields}}(\boldsymbol{Q}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}). \tag{3.11}$$

Theorem 3.1. *All second-order covariant Schrödinger operators*

$$\mathsf{O} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$$

are of the type

$$\mathsf{O}(\Psi) = \alpha \mathsf{S}[\Psi] + \beta r \Psi \qquad \text{with} \quad \alpha, \beta \in \mathbb{C}. \tag{3.12}$$

Proof. Let $\mathfrak{O} : \overset{o}{\text{fields}}(\boldsymbol{Q}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$ be a second-order covariant Schrödinger operator.

(1) By virtue of the 'orbit reduction theorem' [63], implemented by considering the fibred group epimorphism $\mathcal{W}^{(3,2)}(\boldsymbol{Q}) \to \mathcal{W}^{(1,0)}(\boldsymbol{Q})$, we can express $\mathfrak{O}$, through the covariant differentials and curvatures, as $\mathfrak{O}(o, G, K, \mathfrak{h}, \overset{o}{\Upsilon}, \Psi) = \mathfrak{O}'(\nabla^{(1)} o, \nabla^{(2)} G, R[K], \overset{o}{\nabla}{}^{(2)} \mathfrak{h}, R[\overset{o}{\Upsilon}], \overset{o}{\nabla}{}^{(2)} \Psi)$, where the covariant differentials are performed with respect to K and $\overset{o}{\Upsilon}$, as appropriate; we have set

$$\nabla^{(1)} o := (o, \nabla o) \qquad \nabla^{(2)} G := (G, \nabla G, \nabla\nabla G), \ldots \qquad \overset{o}{\nabla}{}^{(2)} \Psi := (\Psi, \overset{o}{\nabla}\Psi, \overset{o}{\nabla}\overset{o}{\nabla}\Psi)$$

the sheaf morphism $\mathfrak{O}'$ invariant with respect to the fibred action of the group bundle $\mathcal{T}(\boldsymbol{E})$ and, for each observer o, the induced sheaf morphism $\overset{o}{\mathfrak{O}}{}'$ is covariant with respect to the fibred actions of the group bundles $\mathcal{W}^{(1,0)}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$.

(2) By considering the fundamental identities, we can express $\mathfrak{O}$, as

$$\mathfrak{O}(o, G, K, \mathfrak{h}, \overset{o}{\Upsilon}, \Psi) = \mathfrak{O}''\big(\nabla^{(1)} o, G, R[K], \mathfrak{h}, o^*\Omega[G, K], \overset{o}{\nabla}{}^{(2)} \Psi\big)$$

where, for each observer o, the induced sheaf morphism $\overset{o}{\mathfrak{O}}{}''$ is covariant with respect to the fibred actions of the group bundles $\mathcal{W}^{(1,0)}(\boldsymbol{Q})$ and $\mathcal{G}(\mathbb{T})$.

(3) Let us consider the Hermitian metric $\widetilde{\mathfrak{h}} := \mathfrak{h}/\eta[G]$, where $\eta[G] : \boldsymbol{E} \to \mathbb{T}^{3/2} \otimes \Lambda^3 V^* \boldsymbol{E}$ is the scaled volume form associated with G.

By virtue of the 'homogeneous function theorem' [63], implemented by considering the subbundle $\boldsymbol{E} \times \mathbb{R}^+ \subset \mathcal{W}^{(1,0)}(\boldsymbol{Q})$ of real homotheties of $\boldsymbol{Q}$, the sheaf $\mathfrak{O}$ can be expressed as a polynomial of the type

$$\mathfrak{O}(o, G, K, \mathfrak{h}, \overset{o}{\Upsilon}, \Psi) = \sum_{\substack{0 \leqslant i, j \leqslant 2 \\ 0 \leqslant h \\ 0 \leqslant k \leqslant 2}} A^{hk}_{ij} \big(\nabla^{(1)} o,\ G, R[K], o^*\Omega[G, K]\big) (\widetilde{\mathfrak{h}}(\overset{o}{\nabla}{}^i \Psi, \overset{o}{\nabla}{}^j \Psi))^h \overset{o}{\nabla}{}^k \Psi$$

where the coefficients A^{hk}_{ij} are $\mathbb{T}^{*(1-3h/2)}$-valued tensors of $\boldsymbol{E}$, which are covariant with respect to $\mathcal{G}^1(\boldsymbol{E})$.

(4) By virtue of the 'homogeneous function theorem' and the 'metric covariant function' [63], implemented by considering the covariance of $\mathfrak{O}$ with respect to the fibred action of the group bundle $\mathcal{G}(\mathbb{T})$, the covariance of A^{hk}_{ij} with respect to the fibred action of the group

bundle $\mathcal{G}^1(\boldsymbol{E})$ and by counting the contravariant and covariant indices of the objects we are dealing with, we can express $\mathfrak{O}$ as

$$\mathfrak{O}(o, G, K, \mathrm{h}, \overset{o}{\mathrm{Ч}}, \Psi) = a\left(C^1_1 \nabla o\right)\Psi + b o \lrcorner \overset{o}{\nabla}\Psi + c\bar{G}(\overset{o}{\nabla}\overset{o}{\nabla}\Psi) + \mathrm{d}r[G]\Psi,$$
$$a, b, c, d \in \mathbb{C}.$$

(5) By virtue of (2.22), the invariance of $\overset{o}{\mathfrak{O}}$ with respect to the fibred action of the group bundle $\mathcal{T}(\boldsymbol{E})$ yields $a = \mathrm{i}c$ and $b = 2\mathrm{i}c$. Hence

$$\begin{aligned}\mathfrak{O}(o, G, K, \mathrm{h}, \overset{o}{\mathrm{Ч}}, \Psi) &= c\left(\left(\mathrm{i}C^1_1 \nabla o\right)\Psi + 2\mathrm{i}o \lrcorner \overset{o}{\nabla}\Psi + \bar{G}(\overset{o}{\nabla}\overset{o}{\nabla}\Psi)\right) + \mathrm{d}r[G]\Psi \\ &= 2\mathrm{i}c\mathrm{S}[\Psi] + \mathrm{d}r[G]\Psi \qquad c, d \in \mathbb{C}\end{aligned}$$

and putting $\alpha = 2\mathrm{i}c$, $\beta = d$ we get the theorem, by observing that $C^1_1 \nabla o = \mathrm{div}_\eta\, o$. □

If our quantum system were involved with a 'fundamental' distinguished timescale, then we could not require the covariance of the Schrödinger operator with respect to the change of timescale. In such a case, step (4) of the above proof would be weaker and we would obtain many more solutions of our problem.

For instance, by postulating a distinguished fundamental timescale $\tau \in \mathbb{T}^*$, any additional term of the type $f \circ (\tau^{*3/2} \otimes \widetilde{\mathrm{h}}(\Psi, \Psi))\tau \otimes \Psi \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$, where $f : \mathbb{C} \to \mathbb{C}$ is any function, would still yield a covariant Schrödinger operator, according to the weakened definition of covariance.

Equivalent considerations hold for a possible distinguished length scale, as timescales and length scales can be related by the fundamental constant $\hbar/m \in \mathbb{T}^* \otimes \mathbb{L}^2$.

3.3.3. Quantum Lagrangian. We prove that all second-order quantum Lagrangians are essentially first-order operators and are proportional to the Lagrangian exhibited in (3.8).

We define a *quantum Lagrangian* to be a sheaf morphism

$$\overset{o}{\mathfrak{O}} : \mathrm{fields}(\boldsymbol{Q}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathrm{i}\Lambda^4 T^*\boldsymbol{E}). \tag{3.13}$$

Theorem 3.2. *All second-order covariant quantum Lagrangians*

$$\mathrm{O} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathrm{i}\Lambda^4 T^*\boldsymbol{E})$$

are of the type

$$\begin{aligned}\mathrm{O}(\Psi) = a\mathrm{L}_{(k)}[\Psi] &+ \tfrac{1}{4}b\,\mathrm{d}t \wedge (\mathrm{h}(\Psi, \mathrm{S}[\Psi]) - \mathrm{h}(\mathrm{S}[\Psi], \Psi)) \\ &+ \tfrac{1}{4}c\mathrm{i}\,\mathrm{d}t \wedge (\mathrm{h}(\mathrm{S}[\Psi], \Psi) + \mathrm{h}(\Psi, \mathrm{S}[\Psi]))\end{aligned} \tag{3.14}$$

with $a, b, c, k \in \mathbb{R}$.

Proof.

(1) By a procedure analogous to that of theorem 3.1, we can prove that all second-order covariant Lagrangians can be expressed as a polynomial of the type

$$\begin{aligned}&\mathfrak{O}(o, G, K, \mathrm{h}, \overset{o}{\mathrm{Ч}}, \Psi) \\ &\quad = \mathrm{i}\sum_{\substack{0\leqslant i,j\leqslant 2\\ 0\leqslant h}} A^h_{ij0}\left(\nabla^{(1)}o, G, R[K], o^*\Omega[G, K]\right)(\widetilde{\mathrm{h}}(\overset{o}{\nabla}{}^i\Psi, \overset{o}{\nabla}{}^j\Psi))^h \upsilon^0\end{aligned}$$

where $\upsilon[G] : \boldsymbol{E} \to \mathbb{T}^{5/2} \otimes \Lambda^4 T^*\boldsymbol{E}$ is the scaled volume form induced by G, and where the coefficients A^h_{ij0} are $\mathbb{T}^{*q}$-valued tensors of $\boldsymbol{E}$, with an appropriate rational number q, and are covariant with respect to $\mathcal{G}^1(\boldsymbol{E})$.

(2) By virtue of the 'homogeneous function theorem' and the 'metric covariant function' [63], implemented by considering the covariance of L with respect to the fibred action of the group bundle $\mathcal{G}(\mathbb{T})$, the covariance of A^h_{ij0} with respect to the fibred action of the group bundle $\mathcal{G}(\boldsymbol{E})$ and by counting the contravariant and covariant indices of the objects we are dealing with, we can express $\mathfrak{O}$ as

$$\begin{aligned}\mathfrak{O}(o, G, K, \mathrm{h}, \overset{o}{\text{Ч}}, \Psi) = {}& \mathrm{i}\alpha \left(C^1_1 \nabla o\right) \mathrm{d}t \wedge \mathrm{h}(\Psi, \Psi) + \mathrm{i}\gamma \,\mathrm{d}t \wedge \bar{G}(\overset{o}{\nabla}\Psi, \overset{o}{\nabla}\Psi) \\ & + \mathrm{i}\epsilon r[G]\,\mathrm{d}t \wedge \mathrm{h}(\Psi, \Psi) + \beta(\mathrm{d}t \wedge \mathrm{h}(o \lrcorner \overset{o}{\nabla}\Psi, \Psi) - \mathrm{d}t \wedge \mathrm{h}(\Psi, o \lrcorner \overset{o}{\nabla}\Psi)) \\ & + \mathrm{i}\beta'(\mathrm{d}t \wedge \mathrm{h}(o \lrcorner \overset{o}{\nabla}\Psi, \Psi) + \mathrm{d}t \wedge \mathrm{h}(\Psi, o \lrcorner \overset{o}{\nabla}\Psi)) + \delta(\mathrm{d}t \wedge \mathrm{h}(\overset{o}{\Delta}\Psi, \Psi) \\ & - \mathrm{d}t \wedge \mathrm{h}(\Psi, \overset{o}{\Delta}\Psi)) + \mathrm{i}\delta'(\mathrm{d}t \wedge \mathrm{h}(\overset{o}{\Delta}\Psi, \Psi) \\ & + \mathrm{d}t \wedge \mathrm{h}(\Psi, \overset{o}{\Delta}\Psi)) \qquad \alpha, \beta, \beta', \gamma, \delta, \delta', \epsilon \in \mathbb{R}.\end{aligned}$$

(3) By virtue of (2.22), the invariance of $\overset{o}{\mathfrak{O}}$ with respect to the fibred action of the group bundle $\mathcal{T}(\boldsymbol{E})$ yields the following identities $\alpha = -2\delta$, $\beta = 2\delta' - \gamma$ and $\beta' = -2\delta$. Hence

$$\begin{aligned}\mathfrak{O}(o, G, K, \mathrm{h}, \overset{o}{\text{Ч}}, \Psi) = {}& -4\gamma \mathrm{L}[\Psi] - \mathrm{i}\epsilon r[G] \otimes \mathrm{d}t \wedge \mathrm{h}(\Psi, \Psi) + 2\mathrm{i}\delta \,\mathrm{d}t \wedge (\mathrm{h}(\mathrm{S}[\Psi], \Psi) \\ & + \mathrm{h}(\Psi, \mathrm{S}[\Psi])) + 2\delta' \,\mathrm{d}t \wedge (\mathrm{h}(\mathrm{S}[\Psi], \Psi) - \mathrm{h}(\Psi, \mathrm{S}[\Psi])) \\ & \gamma, \delta, \delta', \epsilon \in \mathbb{R}.\end{aligned}$$

Then, putting $a = -4\gamma$, $\epsilon = -ka$, $b = -8\delta'$, $c = 8\delta$, we get the theorem. □

We stress that the Euler–Lagrange operator associated with the second-order covariant quantum Lagrangian exhibited in the above theorem is $\mathrm{E}[\mathrm{L}] = (a + b)\mathrm{S}_{(k)}$.

Hence, the second-order terms in the above Lagrangian are not physically relevant.

3.3.4. Schrödinger operator on the functional quantum bundle. We introduce briefly the functional quantum bundle and prove that all covariant Schrödinger operators which induce Hermitian operators on the functional quantum bundle coincide with our distinguished Schrödinger operator.

We define the (infinite-dimensional) *functional quantum bundle* $\boldsymbol{H} \to \boldsymbol{T}$ to be the fibred set over $\boldsymbol{T}$, whose fibres are constituted of the compact support smooth sections, at fixed time, of the quantum bundle ('regular sections').

This bundle has no distinguished splittings into time and type fibre; such a splitting can be obtained by choosing a classical observer.

The quantum metric h equips the functional quantum bundle with a pre-Hilbert metric $\langle\,|\,\rangle$. Then, a true Hilbert bundle can be obtained by a completion procedure.

The functional quantum bundle is naturally endowed with a 'smooth' structure in the sense of Frölicher [40], which allows us to introduce standard geometric notions such as tangent space, connections, etc [46].

Each regular section Ψ of $\boldsymbol{Q}$ can be regarded as a section $\widehat{\Psi}$ of $\boldsymbol{H}$. Accordingly, each 'regular' operator $\mathrm{O} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$ can be regarded as a fibred morphism $\widehat{\mathrm{O}} : \boldsymbol{H} \to \boldsymbol{H}$ over $\boldsymbol{T}$.

In particular, the operator $\widehat{\mathrm{O}}$ associated with a Schrödinger operator can be regarded as the covariant differential of a linear connection of the functional quantum bundle.

A fibred morphism $\widehat{\mathrm{O}} : \boldsymbol{H} \to \boldsymbol{H}$ over $\boldsymbol{T}$ is said to be *Hermitian*, if for each $\hat{\Phi}, \hat{\Psi} \in \sec(\boldsymbol{T}, \boldsymbol{H})$, we have $d\langle\widehat{\Phi} \mid \widehat{\Psi}\rangle = \langle\widehat{\mathrm{O}}(\widehat{\Phi}) \mid \widehat{\Psi}\rangle + \langle\widehat{\Phi} \mid \widehat{\mathrm{O}}(\widehat{\Psi})\rangle$.

Theorem 3.3. *All second-order covariant Schrödinger operators* O*, such that* $\widehat{\mathrm{O}}$ *is Hermitian, are of the type*

$$\mathrm{O}(\Psi) = \mathrm{S}_{(k)}[\Psi] \qquad \textit{with} \quad k \in \mathbb{R}. \tag{3.15}$$

Proof. Let $\widehat{\Phi}, \widehat{\Psi} \in \mathrm{Sec}(\boldsymbol{T}, \boldsymbol{H})$.

By considering the trivialization of $\boldsymbol{H} \to \boldsymbol{T}$ induced by any global observer o, we obtain

$$d\langle \Phi \mid \Psi \rangle = \int L_o(\mathrm{h}(\Phi, \Psi)) = \int \partial_0(\bar{\phi}\psi)\eta + \int \bar{\phi}\psi L_o\eta.$$

Moreover, we have $\langle \widehat{\overset{o}{\nabla}}(\widehat{\Phi}) \mid \widehat{\Psi} \rangle = \langle \widehat{\Phi} \mid \widehat{\overset{o}{\nabla}}(\widehat{\Psi}) \rangle$.

According to theorem 3.1, let us consider the most general second-order covariant Schrödinger operator $\mathrm{O}(\Psi) = \alpha\big(\overset{o}{\nabla}_0 + \frac{1}{2}(\mathrm{div}_\eta\, o)_0 - \mathrm{i}\frac{1}{2}\overset{o}{\Delta}_0\big)\psi u^0 \otimes \mathrm{b} + \beta r \Psi$, with $\alpha, \beta \in \mathbb{C}$. Then, we obtain

$$\begin{aligned}\langle \widehat{\mathrm{O}}(\widehat{\Phi}) \mid \widehat{\Psi} \rangle + \langle \widehat{\Phi} \mid \widehat{\mathrm{O}}(\widehat{\Psi}) \rangle &= \mathrm{re}(\alpha) \int (\partial_0(\bar{\phi}\psi) + \bar{\phi}\psi(\mathrm{div}_\eta\, o)_0)\eta + \mathrm{re}(\beta) \int \bar{\phi}\psi r\eta \\ &\quad + \mathrm{im}(\alpha) \int \left(2A_0\bar{\phi}\psi + \frac{1}{2}(\overline{\overset{o}{\Delta}_0\bar{\phi}})\psi + \frac{1}{2}\bar{\phi}(\overset{o}{\Delta}_0\psi)\right)\eta \\ &= \mathrm{re}(\alpha) d\langle \Phi \mid \Psi \rangle + \mathrm{re}(\beta) \int \bar{\phi}\psi r\eta \\ &\quad + \mathrm{im}(\alpha) \int \left(2A_0\bar{\phi}\psi + \frac{1}{2}(\overline{\overset{o}{\Delta}_0\bar{\phi}})\psi + \frac{1}{2}\bar{\phi}(\overset{o}{\Delta}_0\psi)\right)\eta.\end{aligned}$$

Hence, $\widehat{\mathrm{O}}$ is Hermitian if and only if $\mathrm{re}(\alpha) = 1$, $\mathrm{im}(\alpha) = 0$ and $\mathrm{re}(\beta) = 0$. □

Acknowledgments

This research has been supported by University of Florence, GNFM of INDAM, the Ministry of Educations of the Czech Republic under the project MSM 143100009 and the grant of the GA ČR no 201/02/0225. Thanks are due to H D Döbner for stimulating discussions.

References

[1] Abraham R and Marsden J E 1978 *Foundations of Mechanics* 2nd edn (London: Benjamin)
[2] Balachandran A P, Marmo G, Simoni A and Sparano G 1992 Quantum bundles and their symmetries *Int. J. Mod. Phys.* A **7-8** 1641–67
[3] Bargmann V 1954 *Ann. Math.* **59** 1
[4] Bialynicki-Birula I, Kaliński M and Eberly J H 1994 Lagrange equilibrium points in celestial mechanics and nonspreading wave packets for strongly driven Rydberg electrons *Phys. Rev. Lett.* **73** 77–80
[5] Bialynicki-Birula I and Bialynicki-Birula Z 1997 Rotational frequency shift *Phys. Rev. Lett.* **78** 2539–42
[6] Cabras A 1987 On the universal connection of a system of connections *Note Mat.* **8** 173–209
[7] Cabras A and Kolář I 1995 Connections on some functional bundles *Czech. Math. J.* **45** 529–48
[8] Cabras A and Kolář I 1998 The universal connection of an arbitrary system *Ann. Mat. Pura Appl.* **174** 1–11
[9] Camacho A 1999 On a quantum equivalence principle *Mod. Phys. Lett.* A **14** 275–88
[10] Camacho A 1999 Aharanov–Bohm effect and coordinate transformations *Mod. Phys. Lett.* A **14** 1445–53
[11] Camacho A 1999 Flavor-oscillation clocks, continuous quantum measurements and a violation of Einstein equivalence principle *Mod. Phys. Lett.* A **14** 2545–56
[12] Camacho A 1999 The role of the quantum properties of gravitational radiation in the detection of gravitational waves *Int. J. Mod. Phys.* A **14** 1997–2012
[13] Camacho A 1999 Gravity induced interference and continuous quantum measurement *Phys. Lett.* A **256** 339–46

[14] Camacho A 1999 Measurement induced interference in an inhomogeneous gravitational field *Phys. Lett.* A **262** 110–20

[15] Camacho A and Dehnen H 1999 Higgs field and localization problem *Phys. Lett.* A **256** 339–46

[16] Camacho A and Camacho-Galván A 1999 Time emergence by self-measurement in a quantum anisotropic universe *Nuovo Cimento* B **114** 923–38

[17] Canarutto D, Jadczyk A and Modugno M 1995 Quantum mechanics of a spin particle in a curved spacetime with absolute time *Rep. Math. Phys.* **36** 95–140

[18] Cartan E 1955 *Sur les variétés à connexion affine et la théorie de la rélativité généralisée* (Paris: Gauthier Villars)

[19] Cartan E 1986 *On Manifolds with an Affine Connection and the Theory of General Relativity* (Napoli: Bibliopolis)

[20] Cheng K S 1972 Quantization of a general dynamical system by Feynman's path integration formulation *J. Math. Phys.* **13** 1723–6

[21] de Leon M, Marrero J C and Padron E 1997 On the geometric quantization of Jacobi manifolds *J. Math. Phys.* **38** 6185–213

[22] Dautcourt G 1990 On the Newtonian limit of general relativity *Acta Phys. Pol.* B **21** 755–65

[23] DeWitt B S 1957 Dynamical theory in curved spaces: I. A review of the classical and quantum action principles *Rev. Mod. Phys.* **29** 377–97

[24] Döbner H-D 2002 Remarks on linear quantum theory *Preprint* ASI-TPA, 02/98

[25] Döbner H-D and Goldin G A 1992 On a general nonlinear Schrödinger equation admitting diffusion currents *Phys. Lett.* A **162** 397–401

[26] Döbner H-D and Goldin G A 1994 Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations *J. Phys. A: Math. Gen.* **27** 1771–80

[27] Döbner H-D and Goldin G A 1996 Introducing nonlinear gauge transformations in a family of nonlinear Schrödinger equations *Phys. Rev.* A **54** 3764–71

[28] Döbner H-D, Goldin G A and Nattermann P 1999 Gauge transformations in quantum mechanics and the unification of nonlinear Schrödinger equations *J. Math. Phys.* **40** 49–63

[29] Döbner H-D and Hennig J-D 1997 On a path to nonlinear quantum mechanics *Symmetries in Science IX* ed Gruber and Ramek (New York: Plenum) pp 81–90

[30] Döbner H-D and Nattermann P 1996 Borel quantization: kinematics and dynamics *Acta Phys. Pol.* B **27** 2327–39

[31] Dombrowski H D and Horneffer K 1964 Die differentialgeometrie des Galileischen relativitätsprinzips *Math. Z.* **86** 291–311

[32] Duval C 1985 The Dirac & Levy-Leblond equations and geometric quantization *Differential Geometric Methods in Mathematical Physics (Lecture Notes in Mathematics vol 1251)* ed P L García and A Pérez-Rendón (Berlin: Springer) pp 205–21

[33] Duval C 1993 On Galilean isometries *Class. Quantum Grav.* **10** 2217–21

[34] Duval C, Burdet G, Künzle H P and Perrin M 1985 Bargmann structures and Newton–Cartan theory *Phys. Rev.* D **31** 1841–53

[35] Duval C, Gibbons G and Horvaty P 1991 Celestial mechanics, conformal structures, and gravitational waves *Phys. Rev.* D **43** 3907–21

[36] Duval C and Künzle H P 1984 Minimal gravitational coupling in the Newtonian theory and the covariant Schrödinger equation *Gen. Rel. Grav.* **16** 333–47

[37] Ehlers J 1989 The Newtonian limit of general relativity *Fisica Matematica Classica e Relatività (Elba 9–13 giugno, 1989)* pp 95–106

[38] Fanchi J R 1993 Review of invariant time formulations of relativistic quantum theories *Found. Phys.* **23** 487–548

[39] Feynmann R P 1948 Space-time approach to non-relativistic quantum mechanics *Rev. Mod. Phys.* **20** 267–87

[40] Frölicher A 1982 *Smooth Structures (Lecture Notes in Mathematics vol 962)* (Berlin: Springer) pp 69–81

[41] García P L 1972 Connections and 1-jet fibre bundle *Rendic. Sem. Mat. Univ. Padova* **47** 227–42

[42] García P L 1974 The Poincaré–Cartan invariant in the calculus of variations *Symp. Math.* **14** 219–46

[43] Ghirardi C C, Rimini A and Weber T 1986 Unified dynamics for microscopic and macroscopic systems *Phys. Rev.* D **34** 470–91

[44] Gotay M J 1986 Constraints, reduction and quantization *J. Math. Phys.* **27** 2051–66

[45] Havas P 1964 Four-dimensional formulation of Newtonian mechanics and their relation to the special and general theory of relativity *Rev. Mod. Phys.* **36** 938–65

[46] Jadczyk A, Janyška J and Modugno M 1998 Galilei general relativistic quantum mechanics revisited *Geometria, Física-Matemática e outros Ensaios* ed Homenagem a António Ribeiro Gomes, A S Alves, F J Craveiro de Carvalho and J A Pereira da Silva (Coimbra: University of Coimbra) pp 253–313

[47] Jadczyk A and Modugno M 1992 An outline of a new geometric approach to Galilei general relativistic quantum mechanics *Differential Geometric Methods in Theoretical Physics* ed C N Yang, M L Ge and X W Zhou (Singapore: World Scientific) pp 543–56

[48] Jadczyk A and Modugno M 1994 A scheme for Galilei general relativistic quantum mechanics *General Relativity and Gravitational Physics* ed M Cerdonio, R D'Auria, M Francaviglia and G Magnano (Singapore: World Scientific) pp 319–37

[49] Jadczyk A and Modugno M 1994 Galilei general relativistic quantum mechanics *Report* Department of Applied Mathematics, University of Florence

[50] Janyška J 1995 Remarks on symplectic and contact 2-forms in relativistic theories *Bollettino U.M.I.* B **9** 587–616

[51] Janyška J 1995 Natural quantum Lagrangians in Galilei quantum mechanics *Rendiconti di Matematica (Series VII vol 15)* (Rome: University of Rome) pp 457–68

[52] Janyška J 1998 Natural Lagrangians for quantum structures over 4-dimensional spaces *Rendiconti di Matematica (Series VII vol 18)* (Roma) pp 623–48

[53] Janyška J 2001 A remark on natural quantum Lagrangians and natural generalized Schrödinger operators in Galilei quantum mechanics *Proc. Winter School Geometry and Topology (Srní, 2000) (Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II vol 66)* pp 117–28

[54] Janyška J and Modugno M 1996 Classical particle phase space in general relativity *Proc. Conf. Diff. Geom. and its Appl. (Brno, 1995) (Masaryk University, 1996)* pp 573–602 webpage http://www.emis.de/proceedings

[55] Janyška J and Modugno M 1996 Relations between linear connections on the tangent bundle and connections on the jet bundle of a fibred manifold *Arch. Math. (Brno)* **32** 281–8 webpage http://www.emis.de/journals/

[56] Janyška J and Modugno M 1997 On quantum vector fields in general relativistic quantum mechanics *Proc. 3rd Int. Workshop Differential Geometry and its Applications (Gen. Math.* **5**, *Sibiu University, Romania)* pp 199–217

[57] Janyška J and Modugno M 1998 On the graded Lie algebra of quantisable forms *Differential Geometry and Applications Proc. 7th Int. Conf. (Brno, 10–14 Aug. 1998) (Masaryk University, 1999)* ed I Kolář, O Kowalski, D Krupka and J Slovak pp 601–620 webpage http://www.emis.de/journals

[58] Janyška J and Modugno M 2000 Quantisable functions in general relativity *Opérateurs différentiels et Physique Mathématique (Textos Mat. Ser. B vol 24)* ed J Vaillant and J Carvalho e Silva (Coimbra: University of Coimbra) pp 161–81

[59] Janyška J and Modugno M 2001 Covariant pre-quantum operators *Preprint* (*Proc. Differential Geometry and its Applications Conf. (27–31 August 2001, Silesian University, Opava* to appear)

[60] Janyška J and Modugno M 2001 *Covariant Quantum Mechanics* in preparation

[61] Janyška J, Modugno M and Saller D 2000 Covariant quantum mechanics and quantum symmetries *Proc. Recent Developments in General Relativity (Genova, Sept. 2000)* (Berlin: Springer) at press

[62] Kalinski M and Eberly J H 1996 Trojan wave packets: Mathieu theory and generation from circular states *Phys. Rev.* A **53** 1715–24

[63] Kolář I, Michor P and Slovák J 1993 *Natural Operators in Differential Geometry* (Berlin: Springer)

[64] Krupka D and Janyška J 1990 *Lectures on Differential Invariants* (Brno: Folia Fac. Sci. Nat. Univ. Purkynianae Brunensis)

[65] Kuchař K 1980 Gravitation, geometry and nonrelativistic quantum theory *Phys. Rev.* D **22** 1285–99

[66] Künzle H P 1972 Galilei and Lorentz structures on space-time: comparison of the corresponding geometry and physics *Ann. Inst. H Poincare* **17** 337–62

[67] Künzle H P 1974 Galilei and Lorentz invariance of classical particle interaction *Symp. Math.* **14** 53–84

[68] Künzle H P 1976 Covariant Newtonian limit of Lorentz space-times *Gen. Rel. Grav.* **7** 445–57

[69] Künzle H P 1984 General covariance and minimal gravitational coupling in Newtonian space-time *Geometrodynamics* ed A Prastaro (Bologna: Tecnoprint) pp 37–48

[70] Künzle H P and Duval C 1984 Dirac field on Newtonian space-time *Ann. Inst. H Poincare* **41** 363–84

[71] Kyprianidis A 1987 Scalar time parametrization of relativistic quantum mechanics: the covariant Schrödinger formalism *Phys. Rep.* **155** 1–27

[72] Lämmerzahl C 1996 On the equivalence principle in quantum theory *Gen. Rel. Grav.* **28** 1043–70

[73] Le Bellac M and Levy-Leblond J M 1973 Galilean electromagnetism *Nuovo Cimento* B **14** 217–33

[74] Levy-Leblond J M 1971 Galilei group and Galilean invariance *Group Theory and Its Applications* vol 2 ed E M Loebl (New York: Academic) pp 221–99

[75] Libermann P and Marle Ch M 1987 Symplectic geometry and analytical mechanics (Dordrecht: Reidel)

[76] Mangiarotti L 1979 Mechanics on a Galilean manifold *Riv. Mat. Univ. Parma* **5** 1–14

[77] Mashhoon B 1988 Neutron interferometry in a rotating frame of reference *Phys. Rev. Lett.* **61** 2639–42

[78] Massa E and Pagani E 1991 Classical dynamics of non-holonomic systems: a geometric approach *Ann. Inst. H Poincare* **55** 511–44

[79] Messiah A 1961 *Quantum Mechanics* vols I, II and III (Amsterdam: North-Holland)
[80] Modugno M 1998 On a covariant formulation of quantum mechanics *Proc. 2nd Int. Workshop Lie Theory and Its Applications in Physics (Clausthal, Aug. 17–20, 1997)* ed H-D Doebner, V K Dobrev and J Hilgert (Singapore: World Scientific) pp 183–203
[81] Modugno M, Tejero Prieto C and Vitolo R 2000 Comparison between geometric quantization and covariant quantum mechanics *Proc. Lie Theory and Its Applications in Physics—Lie III (Clausthal, Germany, 11–14 July 1999)* ed H-D Doebner, V K Dobrev and J Hilgert (Singapore: World Scientific) pp 155–75
[82] Modugno M and Vitolo R 1996 Quantum connection and Poincaré–Cartan form *Gravitation, Electromagnetism and Geometrical Structures* ed G Ferrarese (Bologna: Pitagora Editrice) pp 237–79
[83] Pauli W 1958 *Handbuch der Physik* vol 5 ed S Flügge (Berlin: Springer) pp 18–9
[84] Prugovečki E 1992 *Quantum Geometry: A Framework for Quantum General Relativity* (Dordrecht: Kluwer)
[85] Prugovečki E 1995 *Principles of Quantum General Relativity* (Singapore: World Scientific)
[86] Riehle F, Kisters Th, Witte A, Helmecke J and Bordé Ch 1991 Optical Ramsey spectroscopy in a rotating frame: Sagnac effect in a matter–wave interferometer *Phys. Rev. Lett.* **67** 177–80
[87] Saller D and Vitolo R 2000 Symmetries in covariant classical mechanics *J. Math. Phys.* **41** 6824–42
[88] Saunders D J 1989 *The Geometry of Jet Bundles* (Cambridge: Cambridge University Press)
[89] Schmutzer E and Plebanski J 1977 Quantum mechanics in non inertial frames of reference *Fortschr. Phys.* **25** 37–82
[90] Simms D J 1968 *Lie Groups and Quantum Mechanics (Lecture Notes in Mathematics vol 52)* (Berlin: Springer)
[91] Sniatycki J 1980 *Geometric Quantization and Quantum Mechanics* (New York: Springer)
[92] Trautman A 1963 Sur la théorie Newtonienne de la gravitation *C. R. Acad. Sci., Paris II* **257** 617–20
[93] Trautman A 1966 Comparison of Newtonian and relativistic theories of space-time *Perspectives in Geometry and Relativity* vol 42 (Bloomington, IN: Indiana University Press) pp 413–25
[94] Tulczyjew W M 1985 An intrinsic formulation of nonrelativistic analytical mechanics and wave mechanics *J. Geom. Phys.* **2** 93–105
[95] Tuynman G M and Wigerinck W A J J 1987 Central extensions and physics *J. Geom. Phys.* **4** 207–58
[96] Vitolo R 1996 Spherical symmetry in classical and quantum Galilei general relativity *Ann. Inst. H Poincaré* **64** 177–203
[97] Vitolo R 1996 Quantum structures in general relativistic theories *Proc. XII Int. Conf. on G.R. and Grav. Phys. (Roma, 1996)* (Singapore: World Scientific)
[98] Vitolo R 1998 Quantising the rigid body *Proc. VII Conf. on Diff. Geom and Appl. (Brno, 1998)* pp 653–64
[99] Vitolo R 1999 Quantum structures in Galilei general relativity *Ann. Inst. H Poincare* **70** 239–57
[100] Woodhouse N 1992 *Geometric Quantization* 2nd edn (Oxford: Clarendon)